Solution 2019

Problem 1 (17 points) In Parts (a) and (b), you are to compute one integral solution (x_0, y_0, z_0, w_0) of the linear equation

$$30x + 42y + 70z + 105w = 1 \quad \text{where} \quad 30 = 2\cdot3\cdot5,\ 42 = 2\cdot3\cdot7,\ 70 = 2\cdot5\cdot7,\ 105 = 3\cdot5\cdot7. \quad (1)$$

(a, 3pts) First, compute one integral solution (u_0, v_0) to the equation $6u + 35v = 1$. Please show your work.

	35	1	0
	6	0	1
5	5	1	−5
1	1	−1	6

$\underbrace{(-1)}_{v_0} \cdot 35 + \underbrace{6}_{u_0} \cdot 6 = 1$

(b, 6pts) Then compute one integral solution (x_0, y_0) to $30x + 42y = 6u_0$ and one integral solution (z_0, w_0) to $70z + 105w = 35v_0$, with the u_0, v_0 from Part (a); note that by (a) the solution satisfies (1). Please show your work.

$5 \cdot x + 7y = u_0 = 6$

	7	1	0
	5	0	1
1	2	1	−1
2	1	−2	3

$(-2)7 + 3 \cdot 5 = 1$

$\underbrace{-12}_{y_0} \cdot 7 + \underbrace{18}_{x_0} \cdot 5 = 6$

$2z + 3w = v_0 = -1$

	3	1	0
	2	0	1
1	1	1	−1

$1 \cdot 3 + (-1)2 = 1$

$\underbrace{(-1)}_{w_0} 3 + \underbrace{1}_{z_0} \cdot 2 = -1$

(c, 4pts) Please consider the expansion of $(2x+1)^{10}$. What is the coefficient of x^4?

$(2x+1)^{10} = \cdots + \underbrace{\binom{10}{4}(2x)^4}\, 1^6 + \cdots$

$2^4 \binom{10}{4} = 2^4 \binom{10}{6} = 3360$

(d, 4pts) Please determine p_{306}, where p_n is the n-th prime number, e.g., $p_1 = 2$, $p_2 = 3$, $p_{25} = 97$. You may assume that $\pi(2003) = 304$ and $\pi(2020) = 306$.

2003 ~~2005~~ ~~2007~~ div. by 3 ~~2009~~ div by 7 2011 (P_{305})

~~2013~~ div by 3 ~~2015~~ 2017 (P_{306}) ~~2019~~ div. by 3 2021

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Solution 2019

Problem 2 (8 points): Please prove for all integers $a,b \in \mathbb{Z}$, $(a,b) \neq (0,0)$: $\mathrm{GCD}(8a+5b, 5a+3b) = \mathrm{GCD}(a,b)$. [Hint: express both a,b as integer linear combinations of $8a+5b, 5a+3b$.]

	$8a+5b$	$1 \quad 0$
	$5a+3b$	$0 \quad 1$
1	$3a+2b$	$1 \quad -1$
1	$2a+b$	$-1 \quad 2$
1	$a+b$	$2 \quad -3$
1	a	$-3 \quad 5$
1	b	$5 \quad -8$

$(*)\begin{cases} (-3)(8a+5b)+5(5a+3b) = a \\ 5(8a+5b)-8(5a+3b) = b \end{cases}$

$g = \mathrm{GCD}(a,b) \Rightarrow g \mid 8a+5b,\ g \mid 5a+3b$

$\Rightarrow g \mid \mathrm{GCD}(8a+5b, 5a+3b) = d$

But from $(*)$ $d \mid a, d \mid b \Rightarrow d \mid g$

$\Rightarrow d = g$

Problem 3 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 0}$: $\sum_{i=0}^{n}(i^2-i) = \frac{1}{3}(n^3-n)$.

Basis: $n=0$: $0^2-0=0=\frac{1}{3}(0^3-0)$.

Hypo: $\sum_{i=0}^{n}(i^2-i) = \frac{1}{3}(n^3-n)$

Ind. proof: $\sum_{i=0}^{n+1}(i^2-i) = \left(\sum_{i=0}^{n}(i^2-i)\right) + (n+1)^2-(n+1)$

$\underset{\text{Hypo}}{=} \frac{1}{3}(n^3-n) + \frac{1}{3}\underbrace{(3n^2+3n)}_{(n+1)^2-(n+1)}$

$= \frac{1}{3}(\underbrace{n^3+3n^2+3n+1}_{(n+1)^3} - n - 1) = \frac{1}{3}((n+1)^3-(n+1))$

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Problem 4 (6 points): Please place check marks in the following table.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
$\forall n \in \mathbb{Z}_{\geq 2}$: $\binom{2n}{n}$ has a prime factor $> n$.	✓			
There exists a prime twin whose smaller prime is a Mersenne prime and whose larger prime is a Fermat prime.	$2^2-1=3$ $2^2+1=5$ ✓			
The sequence $2^{2^{3n}}+1, n \geq 1$ contains infinitely many primes.				✓
The sequence $2^{3n}-1, n \in \mathbb{Z}_{\geq 1}$ contains infinitely many primes.		✓		
Let $\pi_{4,3}(n)$ be the number of primes $\leq n$ of the form $4k+3, k \in \mathbb{Z}_{\geq 0}$, and $\pi(n)$ be the number of primes $\leq n$. Then $\lim_{n\to\infty} \frac{\pi_{4,3}(n)}{\pi(n)} = \frac{1}{2}$.	✓			
$\forall a,b \in \mathbb{Z}_{\geq 1}$: the sequence $a, a+b, a+2b, a+3b, \ldots, a+10b$ contains at least one composite integer.		✓		

Problem 5 (5 points): Consider complex numbers of the from $a+b\sqrt{-5} \in \mathbb{C}$, where $a,b \in \mathbb{Z}$. Note that such complex numbers have real part a and imaginary part $b\sqrt{5}$. Please prove that there do **not** exist integers $x,y,z,w \in \mathbb{Z}$ such that

$$(x+y\sqrt{-5})(2+0\sqrt{-5})+(z+w\sqrt{-5})(1+\sqrt{-5}) = 1+0\sqrt{-5}.$$

Real part: $2x + z - 5w = 1$

Imag part: $2y\sqrt{5} + z\sqrt{5} + w\sqrt{5} = 0$

$\Rightarrow z = -2y - w$

Plug into real part: $2x - 2y - 6w = 1$ impossible

Second proof: Multiply eqn by $1-\sqrt{-5}$

$\underbrace{2\cdot(x+y\sqrt{-5})(1-\sqrt{-5}) + (z+w\sqrt{-5})\cdot 6}$ $= 1-\sqrt{-5}$

real part is multiple of $2 \neq 1$