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Department of Mathematics

NC STATE UNIVERSITY

MA 410 Theory of Numbers, first mid-semester examination, February 7, 2019
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Your Name: ______________________
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 13 questions, where each question counts for the explicitly given number of points, adding to a total of **44 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **one** 8.5in $\times$ 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

Problem 1 (17 points) In Parts (a) and (b), you are to compute one integral solution (x_0, y_0, z_0, w_0) of the linear equation

$$30x + 42y + 70z + 105w = 1 \quad \text{where} \quad 30 = 2 \cdot 3 \cdot 5,\ 42 = 2 \cdot 3 \cdot 7,\ 70 = 2 \cdot 5 \cdot 7,\ 105 = 3 \cdot 5 \cdot 7. \quad (1)$$

(a, 3pts) First, compute one integral solution (u_0, v_0) to the equation $6u + 35v = 1$. Please show your work.

(b, 6pts) Then compute one integral solution (x_0, y_0) to $30x + 42y = 6u_0$ and one integral solution (z_0, w_0) to $70z + 105w = 35v_0$, with the u_0, v_0 from Part (a); note that by (a) the solution satisfies (1). Please show your work.

(c, 4pts) Please consider the expansion $(2x+1)^{10}$. What is the coefficient of x^4?

(d, 4pts) Please determine p_{306}, where p_n is the n-th prime number, e.g., $p_1 = 2$, $p_2 = 3$, $p_{25} = 97$. You may assume that $\pi(2003) = 304$ and $\pi(2020) = 306$.

Problem 2 (8 points): Please prove for all integers $a, b \in \mathbb{Z}, (a,b) \neq (0,0)$:
$\mathrm{GCD}(8a+5b, 5a+3b) = \mathrm{GCD}(a,b)$.

Problem 3 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 0}$: $\sum_{i=0}^{n} \left(i^2 - i\right) = \frac{1}{3}\left(n^3 - n\right)$.

Problem 4 (6 points): Please place check marks in the following table.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
$\forall n \in \mathbb{Z}_{\geq 2}$: $\binom{2n}{n}$ has a prime factor $> n$.				
There exists a prime twin whose smaller prime is a Mersenne prime and whose larger prime is a Fermat prime.				
The sequence $2^{2^{3n}} + 1, n \geq 1$ contains infinitely many primes.				
The sequence $2^{3n} - 1, n \in \mathbb{Z}_{\geq 1}$ contains infinitely many primes.				
Let $\pi_{4,3}(n)$ be the number of primes $\leq n$ of the form $4k+3, k \in \mathbb{Z}_{\geq 0}$, and $\pi(n)$ be the number of primes $\leq n$. Then $\lim_{n\to\infty} \frac{\pi_{4,3}(n)}{\pi(n)} = \frac{1}{2}$.				
$\forall a, b \in \mathbb{Z}_{\geq 1}$: the sequence $a, a+b, a+2b$, $a+3b$, …, $a+10b$ contains at least one composite integer.				

Problem 5 (5 points): Consider complex numbers of the from $a + b\sqrt{-5} \in \mathbb{C}$, where $a, b \in \mathbb{Z}$. Note that such complex numbers have real part a and imaginary part $b\sqrt{5}$. Please prove that there do **not** exist integers $x, y, z, w \in \mathbb{Z}$ such that

$$\left(x + y\sqrt{-5}\right)\left(2 + 0\sqrt{-5}\right) + \left(z + w\sqrt{-5}\right)\left(1 + \sqrt{-5}\right) = 1 + 0\sqrt{-5}.$$