2018

Problem 1 (18 points) In Parts (a) and (b), you are to compute the integral solution (x,y,z) of a system of 2 simultaneous linear equations:

$$5x+2y-4z=7 \qquad \text{(Eq.1)}$$
$$3x-y-5z=4 \qquad \text{(Eq.2)}$$

Please consider a third equation $11y+13z=1$ (Eq.3) $=3\cdot$(Eq.1) $-5\cdot$(Eq.2).
Please proceed as follows:

(a, 5pts) Please compute the solution (with integer parameter λ) for the diophantine equation (Eq.3) with variables y,z above.

q	r		
	13	1	0
1	11	0	1
5	2	1	-1
2	1	-5	6
	0	11	-13

$(-5)13+6\cdot 11=1$

$y=6-13\lambda$

$z=-5+11\lambda$

(b, 5pts) Please substitute the solution (y,z) for (Eq.3) from Part (a) in terms of λ into (Eq.1) and then solve the equation for the variables x and λ in terms of a new parameter μ.

$5x+2(6-13\lambda)-4(-5+11\lambda)=7$

$5x+12-26\lambda+20-44\lambda=7$

$5x-70\lambda+32=7$

$5x-70\lambda=-25$

q	r		
	70	1	0
14	5	0	1
	0	1	14

$(-5)5+0\cdot\lambda=-25$

$x=-5+14\mu$

$\lambda=\mu$

(c, 4pts) Please consider the coefficients of two trinomial terms (of exponent 11) written as products of binomial coefficients: $\binom{11}{3}\binom{8}{5}$ and $\binom{11}{5}\binom{6}{3}$. Please show that the coefficients are equal.

$\binom{11}{3}\binom{8}{5}=\frac{11!}{3!8!}\cdot\frac{8!}{5!3!}=\frac{11!}{5!3!3!}$

$\binom{11}{5}\binom{6}{3}=\frac{11!}{5!6!}\cdot\frac{6!}{3!3!}=\frac{11!}{5!3!3!}$

(d, 4pts) Please determine $\pi(960)$. You may assume that the $p_{162}=953$, where p_n is the n-th prime number, e.g., $p_1=2$, $p_2=3$, $p_{25}=97$.

$\pi(953)=162$, $\pi(954)=162$, $\pi(955)=162$, $\pi(956)=162$

$3|957$ $\pi(957)=162$, $\pi(958)=162$, $\pi(959)=162$, $7|959$

$\Rightarrow \pi(960)=162$

Problem 2 (8 points): Please prove for all integers $n \in \mathbb{Z}$: $\mathrm{GCD}(n^3+1,\, n^2+n+1) = 1$. [Hint: write 2 as an integer linear combination of the two arguments and show that n^2+n+1 is always odd.]

Let $g = GCD(n^3+1,\, n^2+n+1)$

Then g divides $1\cdot(n^3+1) - (n-1)(n^2+n+1)$

$= n^3+1-(n^3-1) = 2$

So g is either 1 or 2.

However, $n^2+n+1 = n(n+1)+1 =$

an even integer $[n(n+1)]+1 =$ odd,

which g must divide an odd int: $g=1$

Problem 3 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 0}$: $\sum_{i=0}^{n} i^2 2^i = (n^2-2n+3)2^{n+1} - 6$.

$n=0$ $\quad \sum_{i=0}^{0} i^2 \cdot 2^i = \sum_{i=0}^{0} 0 \cdot 2^0 = 0 = (0^2-2\cdot 0+3)\cdot 2 - 6$

Hypothesis for n.

Proof for $n+1$:

$$\sum_{i=0}^{n+1} i^2 2^i = \sum_{i=0}^{n} i^2 2^i + (n+1)^2 2^{n+1}$$

$$\underset{Hypo}{=} (n^2-2n+3)2^{n+1} - 6 + (n^2+2n+1)2^{n+1}$$

$$= (2n^2+4)2^{n+1} - 6 = ((n+1)^2 - 2(n+1)+3)\cdot 2^{n+2}$$

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Problem 4 (6 points): Please place check marks in the following table. ~~Here p_n denotes the n-th prime number~~.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
If a prime number p is a factor of $2^{2^n}+1$ then $\exists k\colon p = k2^{n+2}+1$.	✓			
There exists a prime number p such that $\forall n, p < n < 2p$: n is composite.		✓		
The sequence $k2^{1000}+1, k \in \mathbb{Z}_{\geq 0}$ contains infinitely many primes.	✓			
The number of prime numbers of the form $2^p - 1, p \in \mathbb{Z}_{\geq 2}$ is ≤ 25.		✓		
The number of prime numbers of the form $2^{2^n}+1, n \in \mathbb{Z}_{\geq 0}$ is ≥ 6.				✓
There exist infinitely many prime numbers p such that $p+2$ is a prime number.			✓	

Problem 5 (5 points): True or false: The fundamental theorem of arithmetic remains valid for complex numbers of the form $\alpha + \sqrt{-5}\beta$ where $\alpha, \beta \in \mathbb{Z}$. Please explain.

False: $6 = 2 \cdot 3$

$= (1+\sqrt{-5})(1-\sqrt{-5})$

has 2 prime factorizations.

There is no division with remainder algorithm.