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Department of Mathematics

NC STATE UNIVERSITY

MA 410 Theory of Numbers, first mid-semester examination, February 8, 2018

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Your Name: ______________________

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 13 questions, where each question counts for the explicitly given number of points, adding to a total of **45 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **one** 8.5in $\times$ 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
4	_____
5	_____
Total	_____

Problem 1 (18 points) In Parts (a) and (b), you are to compute the integral solution (x,y,z) of a system of 2 simultaneous linear equations:

$$5x+2y-4z=7 \qquad \text{(Eq.1)}$$
$$3x-y-5z=4 \qquad \text{(Eq.2)}$$

Please consider a third equation $\quad 11y+13z=1 \quad$ (Eq.3) $=3\cdot$(Eq.1) $-\,5\cdot$(Eq.2).
Please proceed as follows:

(a, 5pts) Please compute the solution (with integer parameter λ) for the diophantine equation (Eq.3) with variables y,z above.

(b, 5pts) Please substitute the solution (y,z) for (Eq.3) from Part (a) in terms of λ into (Eq.1) and then solve the equation for the variables x and λ in terms of a new parameter μ.

(c, 4pts) Please consider the coefficients of two trinomial terms (of exponent 11) written as products of binomial coefficients: $\binom{11}{3}\binom{8}{5}$ and $\binom{11}{5}\binom{6}{3}$. Please show that the coefficients are equal.

(d, 4pts) Please determine $\pi(960)$. You may assume that the $p_{162}=953$, where p_n is the n-th prime number, e.g., $p_1=2$, $p_2=3$, $p_{25}=97$.

Problem 2 (8 points): Please prove for all integers $n \in \mathbb{Z}$: $\mathrm{GCD}(n^3+1,\ n^2+n+1) = 1$. [Hint: write 2 as an integer linear combination of the two arguments and show that n^2+n+1 is always odd.]

Problem 3 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 0}$: $\sum_{i=0}^{n} i^2 2^i = (n^2-2n+3)2^{n+1} - 6$.

Problem 4 (6 points): Please place check marks in the following table.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
If a prime number p is a factor of $2^{2^n}+1$ then $\exists k\colon\ p = k2^{n+2}+1$.				
There exists a prime number p such that $\forall n, p < n < 2p\colon\ n$ is composite.				
The sequence $k2^{1000}+1, k \in \mathbb{Z}_{\geq 0}$ contains infinitely many primes.				
The number of prime numbers of the form $2^p - 1, p \in \mathbb{Z}_{\geq 2}$ is ≤ 25.				
The number of prime numbers of the form $2^{2^n}+1, n \in \mathbb{Z}_{\geq 0}$ is ≥ 6.				
There exist infinitely many prime numbers p such that $p+2$ is a prime number.				

Problem 5 (5 points): True or false: The fundamental theorem of arithmetic remains valid for complex numbers of the form $\alpha + \sqrt{-5}\beta$ where $\alpha, \beta \in \mathbb{Z}$. Please explain.