2017

Problem 1 (18 points) You are to compute the integral solution (x,y,z) of a system of 2 simultaneous linear equations:

$$5x-4y+2z=2 \qquad \text{(Eq.1)}$$
$$3x+2y-3z=-15 \qquad \text{(Eq.2)}$$

Please proceed as follows:

(a, 5pts) Please compute the solution (with integer parameters λ and μ) for the diophantine equation (Eq.1) above.

5 1 0
4 0 1
1 1 −1
$5\cdot1-4\cdot1=1$

$5x-4y=1$
$5\cdot u-4\cdot u=u$
$x=u+4\lambda$
$y=u+5\lambda$

$u+2z=2$
$u=-2+2\mu$
$z=2-\mu$

2 1 0
1 0 1
1 1 1 −1
$(-1)+2\cdot1=1$
$-2+2\cdot2=2$

$x=-2+4\lambda+2\mu,\ y=-2+5\lambda+2\mu,\ z=2-\mu$

(b, 5pts) Please substitute the solution for (Eq.1) from Part (a) in terms of λ and μ into (Eq.2) and then solve the equation for λ and μ in terms of a new parameter ν.

$$3(-2+4\lambda+2\mu)+2(-2+5\lambda+2\mu)-3(2-\mu)=-15$$

22 1 0
13 0 1
9 1 1 −1
4 1 −1 2
1 2 3 −5

$22\lambda+13\mu=1$
$22\cdot3+13\cdot(-5)=1$
$\lambda=3+13\nu,\ \mu=-5-22\nu$

(c, 4pts) Please write the trinomial coefficient $\frac{12!}{3!4!5!}$ as a product of two binomial coefficients.

$$\frac{12!}{3!\,9!}\ \frac{9!}{4!\,5!}=\binom{12}{3}\binom{9}{4}$$

(d, 4pts) Please list all prime numbers p with $1038 \le p \le 1050$. You can make use of the fact that $\pi(1038)=174$ and $\pi(1050)=176$. Please show your work.

(1039) 1041 1043 1047 (1049)

1041 div. by 3; 1043 = 7·149; 1047 div by 3

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Problem 2 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 2}$: $\mathrm{GCD}(n^6 - 1,\ n^4 - 1) = n^2 - 1$.

$$n^6 - 1 - n^2(n^4 - 1) = n^2 - 1$$

If $g = GCD(n^6 - 1, n^4 - 1)$ then $g \mid n^2 - 1$

$(n^2 - 1)(n^4 + n^2 + 1) = n^6 - 1 \Rightarrow n^2 - 1 \mid n^6 - 1$

$(n^2 - 1)(n^2 + 1) = n^4 - 1 \Rightarrow n^2 - 1 \mid n^4 - 1$

$\Rightarrow n^2 - 1 \mid g$

Therefore $g = n^2 - 1$

Problem 3 (8 points): Consider the sequence $F_n(x)$ of polynomials in x that is inductively defined for all integers $n \geq 0$ by $F_0(x) = 0$, $F_1(x) = x - 1$ and $F_{n+2}(x) = (x+1)F_{n+1}(x) - xF_n(x)$. Thus the next elements are $F_2(x) = x^2 - 1$, $F_3(x) = x^3 - 1, \ldots$ Please prove by induction that $F_n(x) = x^n - 1$ for all integers $n \geq 0$ (with x^0 defined $= 1$).

Proof by induction

$n = 0$: $F_0(x) = 0 = x^0 - 1 = 1 - 1$

Hypo: $\forall k, 0 \leq k \leq n$: $F_k(x) = x^k - 1$

Ind proof:

Case $n+1 = 1$: $F_1(x) = x - 1 = x^1 - 1$

Case $n+1 \geq 2$: $F_{n+1}(x) = (x+1)F_n(x) - xF_{n-1}(x)$

$$= (x+1)(x^n - 1) - x(x^{n-1} - 1)$$

$$= x^{n+1} - 1$$

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Problem 4 (6 points): Please place check marks in the following table. Here p_n denotes the n-th prime number.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
$\lim_{n\to\infty} \frac{\pi(n)}{\log_e(n)} \geq 1.001$		✓		
There are infinitely many prime numbers whose last 3 decimal digits are 777.	✓			
The sequence $2^{p_n} - 1, n \in \mathbb{Z}_{\geq 1}$ contains infinitely many primes.			✓	✓
$\forall n \geq 2\ \exists p,q$ prime integers: $2n = p+q$			✓	✓
$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$	✓			
There exists an integer g such that the gap between two consecutive primes, $p_{n+1} - p_n$ is equal g infinitely often.	✓			

Problem 5 (5 points): Please state Lagrange's Theorem concerning representing integers as sums-of-squares. Please demonstrate Lagrange's Theorem on the integer 23.

$\forall n \geq 0\ \exists m_1, m_2, m_3, m_4 \in \mathbb{Z}:$

$n = m_1^2 + m_2^2 + m_3^2 + m_4^2$

$23 = 9 + 9 + 4 + 1$