Problem 1 (18 points) You are to compute the integral solution (x, y, z) of a system of 2 simultaneous linear equations:

$$5x - 4y + 2z = 2 (Eq.1)$$

$$3x + 2y - 3z = -15 (Eq.2)$$

Please proceed as follows:

(a, 5pts) Please compute the solution (with integer parameters λ and μ) for the diophantine equation (Eq.1) above.

5 10
$$5x-4y=1$$
 $u+2z=2$ 2 10
4 01 $5\cdot u-4\cdot u=4$ $u=-2+2M$ $(-1)+2\cdot 1=1$
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(b, 5pts) Please substitute the solution for (Eq.1) from Part (a) in terms of λ and μ into (Eq.2) and then solve the equation for λ and μ in terms of a new parameter ν .

$$3(-2+4\lambda+2\mu)+2(-2+5\lambda+2\mu)-3(2-\mu)=-15$$

$$22 \quad |0 \qquad 22\lambda+13\mu=|$$

$$13 \quad |0 \quad |1 \qquad 22\cdot 3+|3 \quad (-5)=|$$

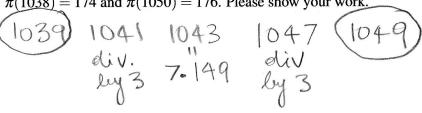
$$4 \quad |-1 \quad 2 \qquad \lambda=3+|3 \quad 0 \quad | \quad \mu=-5-22 \quad \nu$$

$$|2 \quad 3 \quad -5 \quad |2 \quad |$$

(c, 4pts) Please write the trinomial coefficient $\frac{12!}{3!4!5!}$ as a product of two binomial coefficients.

$$\frac{12!}{3!9!} \frac{9!}{4!5!} = \binom{12}{3} \binom{9}{4}$$

(d, 4pts) Please list all prime numbers p with $1038 \le p \le 1050$. You can make use of the fact that $\pi(1038) = 174$ and $\pi(1050) = 176$. Please show your work.



Problem 2 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 2}$: $GCD(n^6 - 1, n^4 - 1) = n^2 - 1$.

$$h^{6}-1 - h^{2}(h^{4}-1) = h^{2}-1$$

If $g = G CD(h^{6}-1, h^{4}-1)$ then $g \mid h^{2}-1$
 $(h^{2}-1)(h^{4}+h^{2}+1) = h^{6}-1 => h^{2}-1 \mid h^{6}-1 >> h^{2}-1 \mid h^{4}-1 >> h^{2}-1 \mid g$

Therefore $g = h^{2}-1$

Problem 3 (8 points): Consider the sequence $F_n(x)$ of polynomials in x that is inductively defined for all integers $n \ge 0$ by $F_0(x) = 0$, $F_1(x) = x - 1$ and $F_{n+2}(x) = (x+1)F_{n+1}(x) - xF_n(x)$. Thus the next elements are $F_2(x) = x^2 - 1$, $F_3(x) = x^3 - 1$,... Please prove by induction that $F_n(x) = x^n - 1$ for all integers $n \ge 0$ (with x^0 defined = 1).

Proof by induction

$$N=0$$
: $F_0(x)=0=x^0-1=1-1$
Hypo: $\forall k, 0 \in k \le n : F_k(x)=x^0-1$
Ind proof:
Case $n+1=1$: $F_1(x)=x-1=x^0-1$
Couse $n+1 \ge 2$: $F_{n+1}(x)=(x+1)F_n(x)-xF_{n+1}(x)$
 $=(x+1)(x^n-1)-x(x^{n-1}-1)$
 $=x^{n+1}-1$

Problem 4 (6 points): Please place check marks in the following table. Here p_n denotes the n-th

prime number.				
Statement	Proved to	Proved to	conjectured	conjectured
	be true	be false	to be true	to be false
$\lim_{n \to \infty} \frac{\pi(n)}{\log_e(n)} \ge 1.001$				
There are infinitely many prime numbers whose last 3 decimal digits are 777.	V			
The sequence $2^{p_n} - 1, n \in \mathbb{Z}_{\geq 1}$ contains infinitely many primes.	, ,		/	
$\forall n \geq 2 \ \exists p, q \text{ prime integers: } 2n = p + q$				V
$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$	V			
There exists an integer g such that the gap between two consecutive primes, $p_{n+1} - p_n$ is equal g infinitely often.	V			

Problem 5 (5 points): Please state Lagrange's Theorem concerning representing integers as sumsof-squares. Please demonstrate Lagrange's Theorem on the integer 23.