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Department of Mathematics

NC STATE UNIVERSITY

MA 410 Theory of Numbers, first mid-semester examination, February 9, 2017

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Your Name: ________________________

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 10 questions, where each question counts for the explicitly given number of points, adding to a total of **45 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **one** 8.5in $\times$ 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

Problem 1 (18 points) You are to compute the integral solution (x, y, z) of a system of 2 simultaneous linear equations:

$$5x - 4y + 2z = 2 \qquad \text{(Eq.1)}$$
$$3x + 2y - 3z = -15 \qquad \text{(Eq.2)}$$

Please proceed as follows:

(a, 5pts) Please compute the solution (with integer parameters λ and μ) for the diophantine equation (Eq.1) above.

(b, 5pts) Please substitute the solution for (Eq.1) from Part (a) in terms of λ and μ into (Eq.2) and then solve the equation for λ and μ in terms of a new parameter ν.

(c, 4pts) Please write the trinomial coefficient $\dfrac{12!}{3!4!5!}$ as a product of two binomial coefficients.

(d, 4pts) Please list all prime numbers p with $1038 \leq p \leq 1050$. You can make use of the fact that $\pi(1038) = 174$ and $\pi(1050) = 176$. Please show your work.

Problem 2 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 2}$: $\mathrm{GCD}(n^6 - 1,\ n^4 - 1) = n^2 - 1$.

Problem 3 (8 points): Consider the sequence $F_n(x)$ of polynomials in x that is inductively defined for all integers $n \geq 0$ by $F_0(x) = 0$, $F_1(x) = x - 1$ and $F_{n+2}(x) = (x+1)\,F_{n+1}(x) - x\,F_n(x)$. Thus the next elements are $F_2(x) = x^2 - 1$, $F_3(x) = x^3 - 1, \ldots$ Please prove by induction that $F_n(x) = x^n - 1$ for all integers $n \geq 0$ (with x^0 defined $= 1$).

Problem 4 (6 points): Please place check marks in the following table. Here p_n denotes the n-th prime number.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
$\lim_{n\to\infty} \frac{\pi(n)}{n/\log_e(n)} \geq 1.001$				
There are infinitely many prime numbers whose last 3 decimal digits are 777.				
The sequence $2^{p_n} - 1, n \in \mathbb{Z}_{\geq 1}$ contains infinitely many primes.				
$\forall n \geq 2\ \exists p, q$ prime integers: $2n = p + q$				
$\sum_{n=1}^{\infty} \frac{1}{p_n} = \infty$				
There exists an integer g such that the gap between two consecutive primes, $p_{n+1} - p_n$ is equal g infinitely often.				

Problem 5 (5 points): Please state Lagrange's Theorem concerning representing integers as sums-of-squares. Please demonstrate Lagrange's Theorem on the integer 23.