2016

Problem 1 (19 points)

(a, 2pts) Please give the complete integral solution (with an integer parameter λ_1) for the diophantine linear equation $4u+3v=1$ in the integer variables u and v. Please show your work.

4	1	0
3	0	1
1	1	−1

$1\cdot 4+(-3)\cdot 3=1$

$u=1+3\lambda_1$

$v=-1-4\lambda_1$

(b, 9pts) b1. Please give the complete integral solution (with the integer parameters u, λ_2) for the diophantine linear equation $12x+8y=4u$ in the integer variables x and y.

12	1	0
8	0	1
4	1	−1

$1\cdot 12+(-1)\cdot 8=4$

$x=u+2\lambda_2$

$y=-u-3\lambda_2$

b2. Please give the complete integral solution (with the integer parameters v, λ_3) for the diophantine linear equation $9z+6w=3v$ in the integer variables z and w.

9	1	0
6	0	1
3	1	−1

$1\cdot 9+(-1)\cdot 6=3$

$z=v+2\lambda_3$

$w=-v-3\lambda_3$

b3. Then substitute the solution from Part (a): for u in (b1) and for v in (b2), to obtain the complete integer solution (with the integer parameters $\lambda_1, \lambda_2, \lambda_3$) for the diophantine linear equation $\underbrace{12x+8y}_{4u}+\underbrace{9z+6w}_{3v}=1$ in the integer variables x, y, z, w. Please show your work.

$x = 1+3\lambda_1+2\lambda_2$

$y = -1-3\lambda_1-3\lambda_2$

$z = -1-4\lambda_1+2\lambda_3$

$w = 1+4\lambda_1-3\lambda_3$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \lambda_1 \begin{bmatrix} 3 \\ -3 \\ -4 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}$$

(c, 4pts) Please find two Gaussian (complex) integers α, β such that $\alpha, \beta \notin \{1,-1,i,-i\}$ ($i=\sqrt{-1}$) and $\alpha\beta=37$.

$\alpha = 6+i$

$\beta = 6-i$

(d, 4pts) Please compute $\pi(127)$. You may assume that $p_{30}=113$ (the 30-th prime number).

113, 115, 117, 119, 121, 123, 125, 127 $=p_{31}$

divisor: 5, 3, 7, 11, 3, 5, prime

−1 −2

$\pi(127)=31$

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Problem 2 (8 points): Please prove for all integers $n \in \mathbb{Z}$:

$$\mathrm{GCD}(n^3+2n-3,\ n^2+1) = \mathrm{GCD}(n^2+1, n-3),$$

where both GCD's are defined because $n^2+1 \neq 0$.

$$n^3+2n-3 - n(n^2+1) = n-3 \qquad (1)$$

$$n^3+2n-3 = n(n^2+1) + (n-3) \qquad (2)$$

Let $g_1 = GCD(n^3+2n-3, n^2+1)$ and $g_2 = GCD(n^2+1, n-3)$.

By (1) $g_1 \mid n-3$ so $g_1 \mid g_2$

By (2) $g_2 \mid n^3+2n-3$ so $g_2 \mid g_1$

Hence $g_1 = g_2$

Problem 3 (8 points): Consider the sequence $F_n(x)$ of polynomials in x that is inductively defined for all integers $n \geq 0$ by $F_0(x) = 0$, $F_1(x) = x$ and $F_{n+2}(x) = 2xF_{n+1}(x) - x^2F_n(x)$. Thus the next elements are $F_2(x) = 2x^2$, $F_3(x) = 3x^3, \ldots$ Please prove by induction that $F_n(x) = nx^n$ for all integers $n \geq 0$ (with x^0 defined $= 1$).

Basis: $n=0 \quad F_0(x) = 0 = 0 \cdot x^0$

$n = 1 \quad F_1(x) = x = 1 \cdot x^1$

Hypo: $F_k(x) = kx^k$ for all $0 \leq k \leq n$, $n \geq 1$

Ind proof: $F_{n+1}(x) = 2x F_n(x) - x^2 F_{n-1}(x)$

$$\underset{(n\geq 1)}{=} 2x\, n \cdot x^n - x^2 (n-1) x^{n-1}$$

$$= 2n x^{n+1} - (n-1) x^{n+1} = (n+1) x^{n+1}$$

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Problem 4 (6 points): Please place check marks in the following table.

Statement	Proved to be true	Proved to be false	conjectured to be true	conjectured to be false
The sequence $n^2+1, n \in \mathbb{Z}_{\geq 1}$ contains infinitely many primes.			✓	
The sequence $100n+99, n \in \mathbb{Z}_{\geq 0}$ contains infinitely many primes.	✓			
The sequence $2^{2^n}+1, n \in \mathbb{Z}_{\geq 0}$ contains infinitely many primes.				✓
There exist $d>0, p$ such that all $p, p+d, p+2d, p+3d, \ldots, p+1000\,d$ are prime.	✓			
The gap between two consecutive primes, $p_{n+1}-p_n$ can be arbitrarily large, that is, $\forall G \geq 2\ \exists n \geq 1 : p_{n+1}-p_n > G$.	✓			
The gap between two consecutive primes, $p_{n+1}-p_n$ is equal 2 infinitely often.			✓	

Problem 5 (5 points): Please state the theorem on prime numbers whose proof was first given by Charles de la Vallée Poussin. Please also give the name of the second number theorist who proved it at the same time independently.

Prime Number Theorem: $\lim_{n \to \infty} \frac{\pi(n)}{n/\log_e(n)} = 1$

J. HADAMARD

J. H. OK