Problem 1 (18 points)

(a, 5pts) Please give the complete integral solution (with an integer parameter λ) for the diophantine linear equation $2u + 15z = 10$ in the integer variables u and z. Please show your work.

	15	1	0
	2	0	1
7	1	1	−7

$1 \cdot 15 - 7 \cdot 2 = 1$

$\underbrace{10}_{z_0} \cdot 15 - \underbrace{70}_{u_0} \cdot 2 = 10$

$u = -70 + 15\lambda$

$z = 10 - 2\lambda$

15λ vs. $\lambda 15$

$z_0 = -2$, 2

$u_0 = 20$, -10

OK

no λ -3

(b, 5pts) Please give the complete integral solution (with the integer parameters u, μ) for the diophantine linear equation $6x + 10y = 2u$ in the integer variables x and y.
Then substitute the solution from Part (a) for u, z to obtain the complete integer solution for the diophantine linear equation $6x + 10y + 15z = 10$ in the integer variables x, y, z. Please show your work.

$x =$ $(2+5\mu)u$

$y =$ ~~$(-1-3\mu)u$~~

	10	1	0
	6	0	1
	~~4~~	~~1~~	~~−1~~
−2 1	2	−1	2
2	0		

$6 \cdot 2 + 10(-1) = 2$

$x_0 = -3u$

$y_0 = 2u$

OK

$x_0 = 2u$ $\quad x = 2u + 5\mu$

$y_0 = -u$ $\quad y = -u - 3\mu$

$\pm 10\mu$

$\mp 6\mu$

-2

$x = -140 + 30\lambda + 5\mu$

$y = 70 - 15\lambda - 3\mu$

$z = 10 - 2\lambda$

no μ

(c, 4pts) Please write the rational number $\frac{90}{55}$ as a (finite) continued fraction $q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ddots}}$.

	90
	55
1	35
1	20
1	15
1	5
3	0

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{3}}}}$$

$1 + \frac{1}{11/7}$ $+1$

(d, 4pts) Please list all prime numbers p with $1010 \le p \le 1020$. You can make use of the fact that $\pi(1010) = 169$ and $\pi(1020) = 171$. Please show your work.

2 primes in range: candidates: ~~1011~~, 1013, ~~1017~~, 1019

div. by 3   div. by 3

Answer: 1013, 1019

$$\underbrace{7a(11a-12)-11a(7a-8)}_{4a},-2(2a+1) = -2 \quad (2a-1)(2a+1)$$

$$7a^2(11a-12)-11a^2(7a-8)-2a(2a+1)+2a+1=1$$

Problem 2 (8 points): Please prove for all integers $a \in \mathbb{Z}$: $\mathrm{GCD}(11a-12, 7a-8, 2a+1) = 1$.

$$7(11a-12)-11(7a-8) = -84+88 = 4$$

Therefore $g = \mathrm{GCD}(11a-12, 7a-8)$ divides 4

Hence $\mathrm{GCD}(g, 2a+1) = 1$ because $2a+1$ is an odd integer.

$$\exists s,t,u: \quad s(11a-12)+t(7a-8)+u(2a+1)=1$$

$-4 \quad 6 \quad 1 \quad +2$

Problem 3 (8 points): Let $k \in \mathbb{Z}_{\geq 0}$ be fixed. Please prove for all integers $n \in \mathbb{Z}_{\geq 0}$ the following identity for binomial coefficients:

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{k+n}{k} = \binom{k+n+1}{k+1}.$$

$$n=0: \quad \sum_{i=0}^{0} \binom{k+i}{k} = \binom{k}{k} = \binom{k+1}{k+1} = 1$$

$$\text{Hypothesis: } \forall 0 \leq m \leq n: \quad \sum_{i=0}^{m} \binom{k+i}{k} = \binom{k+m+1}{k+1}$$

$$\text{Ind. proof: } \sum_{i=0}^{n+1} \binom{k+i}{k} = \left(\sum_{i=0}^{n} \binom{k+i}{k}\right) + \binom{k+n+1}{k}$$

$$\underset{\text{Hypo}}{=} \binom{k+n+1}{k+1} + \binom{k+n+1}{k} \underset{\text{Pascal's id.}}{=} \binom{k+n+2}{k+1}$$

Problem 4 (5 points): Please state **three** proven theorems on the prime numbers.

$\pi(n) \approx n / \log_e(n)$

Dirichlet

Green - Tao

Chebyshev

$p_n^2 < p_1 p_2 \cdots p_{n-1}$ for $n \geq 5$

state a conjecture -1

~~[illegible]~~ fundamental theorem of arith twice -1

2 theorems -1

Problem 5 (5 points): Please state **three** unproven conjectures on the prime numbers.

Goldbach

Twin

Fermat primes

Mersenne primes

State a theorem -1

only state name of conj. OK

Goldbach for odd ~~[illegible]~~ no penalty

1 conj. -3