

**Problem 1** (18 points)

(a, 5pts) Please give the complete integral solution (with an integer parameter  $\lambda$ ) for the diophantine linear equation  $2u + 15z = 10$  in the integer variables  $u$  and  $z$ . Please show your work.

$$\begin{array}{rrr} 15 & 1 & 0 \\ 2 & 0 & 1 \\ 7 & 1 & -1 \end{array}$$

$$\begin{array}{l} 1 \cdot 15 - 7 \cdot 2 = 1 \\ \cancel{10}, \cancel{15} - \cancel{7} \cdot 2 = 10 \end{array}$$

$$u = -70 + 15\lambda$$

$$z = 10 - 2\lambda$$

$$15\lambda \text{ vs. } 15$$

$$z_0 = -2, \cancel{+2}$$

$$u_0 = 20, \cancel{-10}$$

OK

$$\text{no } \lambda = -3$$

(b, 5pts) Please give the complete integral solution (with the integer parameters  $u, \mu$ ) for the diophantine linear equation  $6x + 10y = 2u$  in the integer variables  $x$  and  $y$ .

Then substitute the solution from Part (a) for  $u, z$  to obtain the complete integer solution for the diophantine linear equation  $6x + 10y + 15z = 10$  in the integer variables  $x, y, z$ . Please show your work.

$$x_0 =$$

$$(2+5\mu)u \quad 10 \quad 1 \quad 0$$

$$y = \quad 6 \quad 0 \quad 1$$

$$(-1-3\mu)u \quad 4 \quad 1 \quad -1$$

$$-2 \quad 1 \quad 2 \quad -1 \quad 2$$

$$2 \quad 0$$

$$x_0 = -3u$$

$$y_0 = 2u$$

OK

$$x_0 = 2u$$

$$x = 2u + 5\mu$$

$$y_0 = -u$$

$$y = -u - 3\mu$$

$$\begin{aligned} x &= -140 + 30\lambda + 5\mu \\ y &= 70 - 15\lambda - 3\mu \\ z &= 10 - 2\lambda \end{aligned}$$

(c, 4pts) Please write the rational number  $\frac{90}{55}$  as a (finite) continued fraction  $q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \dots}}$ .

no  $\mu$

$$\begin{array}{r} 90 \\ 55 \\ | \\ 35 \\ | \\ 20 \\ | \\ 15 \\ | \\ 5 \\ | \\ 3 \quad 0 \end{array}$$

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}}}$$

$$1 + \cfrac{1}{11} + 1$$

(d, 4pts) Please list all prime numbers  $p$  with  $1010 \leq p \leq 1020$ . You can make use of the fact that  $\pi(1010) = 169$  and  $\pi(1020) = 171$ . Please show your work.

div. by

2 primes in range: candidates:  $1011, 1013, 1017, \cancel{1015}$

div.  
by 3

$1019$

Answer:  $1013, 1019$

$$\underbrace{7a(11a-12) - 11a(7a-8)}_{4a} - 2(2a+1) = -2 \frac{(2a-1)(2a+1)}{(2a-1)(2a+1)}$$

$$7a^2(11a-12) - 11a^2(7a-8) - 2a(2a+1) + 2a+1 = 1$$

Problem 2 (8 points): Please prove for all integers  $a \in \mathbb{Z}$ :  $\text{GCD}(11a-12, 7a-8, 2a+1) = 1$ .

$$7(11a-12) - 11(7a-8) = -84 + 88 = 4$$

Therefore  $g = \text{GCD}(11a-12, 7a-8)$  divides 4

Hence  $\text{GCD}(g, 2a+1) = 1$  because  $2a+1$  is an odd integer.

$$\exists s, t, u: \begin{matrix} s(11a-12) + t(7a-8) + u(2a+1) = 1 \\ -4 \qquad \qquad \qquad 6 \qquad \qquad \qquad 1 \qquad +2 \end{matrix}$$

Problem 3 (8 points): Let  $k \in \mathbb{Z}_{\geq 0}$  be fixed. Please prove for all integers  $n \in \mathbb{Z}_{\geq 0}$  the following identity for binomial coefficients:

$$\sum_{i=0}^n \binom{k+i}{k} = \binom{k}{k} + \binom{k+1}{k} + \dots + \binom{k+n}{k} = \binom{k+n+1}{k+1}.$$

$$n=0: \sum_{i=0}^0 \binom{k+i}{k} = \binom{k}{k} = \binom{k+1}{k+1} = 1$$

$$\text{Hypothesis: } \forall 0 \leq m \leq n: \sum_{i=0}^m \binom{k+i}{k} = \binom{k+m+1}{k+1}$$

$$\text{Ind. Proof: } \sum_{i=0}^{n+1} \binom{k+i}{k} = \left( \sum_{i=0}^n \binom{k+i}{k} \right) + \binom{k+n+1}{k}$$

$$\stackrel{\text{Hypo}}{=} \binom{k+n+1}{k+1} + \binom{k+n+1}{k} = \binom{k+n+2}{k+1} \quad \text{Pascal's id.}$$

Problem 4 (5 points): Please state three proven theorems on the prime numbers.

$$\pi(n) \approx n / \log_e(n)$$

Dirichlet

Please do not write your name on the subsequent pages.

This exam consists of 5 problems, which are subdivided into 8 questions, where each question counts for the explicitly given number of points, adding up to a total of 50 points. You may write your answers in the space provided, or below the questions, using the back of the sheets for computations. Show all work, if necessary. You are allowed to consider one 8.5in x 11in sheet with notes, but not your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

$$p_n^2 < p_1 p_2 \cdots p_{n-1}$$

for  $n \geq 5$

You will have 75 minutes to do this test.

Good luck!

State a conjecture - 1

fundamental theorem of arithmetic twice - 1

2 theorems - 1

Problem 5 (5 points): Please state three unproven conjectures on the prime numbers.

Goldbach

Twin

Fermat primes

Mersenne primes

State a theorem - 1

only state name of conj. OK

Goldbach for odd

# no penalty

1 conj. - 3