Problem 1 (18 points)

(a, 5pts) Please give the solution (with an integer parameter λ) for the diophantine equation $315x + 217y = 14$ in the integer variables x and y. Please show your work.

315		1	0
217	1	0	1
98	1	1	-1
21	2	-2	3
14	4	9	-13 1pt
3pts 7	1	-11	16
0	2	31	-45

$(-11)\,315 + 16 \cdot 217 = 7$

$x = 2\cdot(-11) + \lambda \frac{217}{7}$ 1pt

$y = 2 \cdot 16 - \lambda \frac{315}{7}$ 1pt

$-22 + \lambda \cdot 31$

$32 - \lambda \cdot 45$

$+217\lambda$ no penalty

-315λ

(b, 5pts) Please consider the expansion of the trinomial $(x+y+z)^6$. What is the multinomial coefficient of the term $x^2y^2z^2$ in that expansion?

$$\frac{6!}{2!2!2!} = \frac{6\cdot5\cdot\cancel{4}\cdot3\cdot\cancel{2}}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 90$$

$\binom{6}{2}\cdot\binom{4}{2}$ no penalty

15 6

(c, 4pts) Please list 5 prime numbers of the form n^2+1, where $n \in \mathbb{Z}_{\geq 1}$.

2, 5, 17, 37, 101, $14^2+1 = 197$, $2^{2^3}+1 = 257 = F_3$

(d, 4pts) From the fact that $p_{55} = 257$, where p_{55} is the 55-th prime number and $p_1 = 2, p_2 = 3, p_3 = 5$, …, deduce the value of $\pi(262)$. Please show your work.

$2 \mid 258$, $7 \mid 259$, $2 \mid 260$, $3 \mid 261$, $2 \mid 262$

$\pi(262) = \pi(257) = 55$

259 prime −1

$\frac{262}{\log_e 262}$

+1

Problem 2 (8 points): Please prove for all integers $a,b \in \mathbb{Z}$: $\mathrm{GCD}(a+2b-1, 2a+b+2, b-1) = 1$. [Hint: find Bezout coefficients; you can eliminate b from the first and second argument using the third.]

$$a+2b-1 \quad -2(b-1) \quad = \quad a+1$$

$$2a+b+2 \quad -(b-1) \quad = \quad 2a+3$$

$$2a+3-2(a+1) = 1$$

$$(-2)(a+2b-1) + (2a+b+2) + 3(b-1) = 1$$

Problem 3 (8 points): Please prove for all integers $n \in \mathbb{Z}_{\geq 2}$:

$$\sum_{i=1}^{n-1} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = 1 - \frac{1}{n}.$$

$n=2$: $\quad \frac{1}{1\cdot 2} = 1 - \frac{1}{2}$

$n+1$ $\quad \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n-1} \frac{1}{i(i+1)} + \frac{1}{n(n+1)}$ 4pts

$$\underset{\text{Hypo}}{=} 1 - \frac{1}{n} + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

4pts

Problem 4 (5 points): True or false: for all p that are prime numbers and for all $i \in \mathbb{Z}$ with $1 \leq i \leq p-1$ the binomial coefficient $\binom{p}{i}$ is divisible by p. Please explain. [Hint: consider the factorial representation of the binomial coefficient.]

Since $\binom{p}{i} = \frac{p!}{i!(p-i)!} \in \mathbb{Z}$

$i!(p-i)! \mid p!$ 3pts

But $GCD(i!(p-i)!, p) = 1$ as p is not a prime factor of $i!(p-i)!$. So

$i!(p-i)! \mid (p-1)!$ and $\binom{p}{i} = p \cdot \frac{(p-1)!}{i!(p-i)!}$

True. 2pts $\binom{p}{i} \neq p\binom{p-1}{i}$

Problem 5 (5 points): True or false: there are infinitely many prime numbers whose decimal representation ends with 001, i.e., are of the form $1000k+1$ for integers k. Please explain.

True. 2pts 3pts

$GCD(1000, 1) = 1$ so by Dirichlet's Theorem there are inf. many primes in the arithm. progr. $1000k+1$