2011

Problem 1 (18 points)

(a, 5pts) Please give the solution (with an integer parameter λ) for the diophantine equation $138x + 384y = 18$ in the integer variables x and y. Please show your work.

384		1	0
138		0	1
108	2	1	−2
30	1	−1	3
18	3	4	−11
12	1	−5	14
6	1	9	−25
0	2	−23	64

$9 \cdot 384 + (-25)138 = 6$

$27 \cdot 384 + (-75)138 = 18$

4 −11

$x = -75 + 64\lambda$

$y = 27 - 23\lambda$

(b, 4pts) Please list the first 4 Mersenne prime numbers.

$2^2 - 1 = 3$

$2^3 - 1 = 7$

$2^5 - 1 = 31$

$2^7 - 1 = 127$

(c, 5pts) Consider the factorization into primes of 100!, namely $100! = 2^{97}3^{48}5^{24}\ldots$.[1] How many distinct primes occur in the full factorization, and which primes occur once, i.e., have exponent 1? Please explain.

primes: $\pi(100) = 25$

Single primes: 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

(d, 4pts) Please factor 29 in the Gaussian integers $\mathbb{G} = \{a + i\,b \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$, where $i = \sqrt{-1}$.

$29 = (2+5i)(2-5i)$

[1] By $n!$ we denote the factorial, $n! = 1 \cdot 2 \cdots (n-1) \cdot n$.

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Problem 2 (8 points): Please prove for all integers $n \geq 0$:

$$\sum_{i=0}^{n} 2^{n-i}\binom{n}{n-i} = 2^n\binom{n}{n} + 2^{n-1}\binom{n}{n-1} + \cdots + 2\binom{n}{1} + \binom{n}{0} = 3^n.$$

$$(x+y)^n = \sum_{i=0}^{n} x^{n-i} y^i \binom{n}{i}$$

Set $x=1,\ y=2$: $\quad 3^n = \sum_{i=0}^{n} 2^i \binom{n}{i}$

$$= 2^n\binom{n}{n} + 2^{n-1}\binom{n}{n-1} + \cdots 2^0\binom{n}{0}$$

Problem 3 (8 points): Consider the sequence a_n of rational numbers that is inductively defined for all integers $n \geq 0$ by $a_0 = 0$, $a_1 = 1/2$ and $a_{n+2} = a_{n+1} - \frac{1}{4}a_n$. Thus the next elements are $a_2 = 1/2$, $a_3 = 3/8$, $a_4 = 1/4, \ldots$ Please prove by induction that $a_n = n2^{-n}$ for all integers $n \geq 0$.

Basis: $n=0$: $a_0 = 0 = 0 \cdot 2^0$

$n=1$: $a_1 = \frac{1}{2} = 1 \cdot 2^{-1}$

Hypo: $\forall i : 0 \leq i \leq n : a_i = i\, 2^{-i}$

Ind. proof for $n+1 \geq 2$

$$a_{n+1} = a_n - \frac{1}{4} a_{n-1}$$

$$= n\,2^{-n} - \frac{1}{4}(n-1)\,2^{-(n-1)}$$

$$= 2^{-(n+1)}\left(2n - (n-1)\right) = (n+1)\,2^{-(n+1)}$$

$$2^{n-i}\binom{n}{n-i} = \left[\binom{n-1}{n-i-1} + \binom{n-1}{n-i}\right]2^{n-i}$$

$$= 2 \cdot 2^{n-i-1}\binom{n-1}{n-i-1} + 2^{n-i}\binom{n-1}{n-i}$$

$$\sum_{i=0}^{n} 2^{n-i}\binom{n}{n-i}$$

$$= \sum_{i=-\infty}^{\infty} 2^{n-i}\binom{n}{n-i} \qquad \left[\binom{n}{i} = 0 \text{ for } i<0 \text{ and } i>n\right]$$

$$= \sum_{i=-\infty}^{\infty}\left[2 \cdot 2^{n-i-1}\binom{n-1}{n-i-1} + 2^{n-i}\binom{n-1}{n-i}\right]$$

$$= 2 \cdot \sum_{i=-\infty}^{\infty} 2^{n-i-1}\binom{n-1}{n-i-1} + \sum_{i=-\infty}^{\infty} 2^{n-i}\binom{n-1}{n-i}$$

$$= 2 \cdot 3^{n-1} + 3^{n-1} = 3^{n}$$

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Problem 4 (5 points): True or false: for all integers $n \geq 2$ and all integers i with $2 \leq i \leq n$, none of the integers $n! + i$ are prime numbers. Please explain.

Because i divides $n!$, $\frac{n!}{i}$ is an integer. Therefore $n! + i = i(\frac{n!}{i} + 1)$ is divisible by i.

Problem 5 (5 points): True or false: for all integers $n \geq 2$ and all integers i with $2 \leq i \leq n!$, none of the integers $n! + i$ are prime numbers. Please explain. [Hint: Chebyshev.]

$\forall m \geq 4 \; \exists p$ prime $m < p < 2m-2$

False: $n = 3$, $i = 5$: $2 < 5 < 6 = 3!$ $3! + 5 = 11$ prime

Let $m = n! + 1 \geq 4 \quad (n \geq 3)$

Then $2m - 2 = 2(n!+1) - 2$

$= 2(n!)$

So there is a prime p with $n! + 1 < p < 2n!$

The statement is only true for $n=2$.