North Carolina State University is a land-
grant university and a constituent institution
of The University of North Carolina

Department of Mathematics

NC STATE UNIVERSITY

MA 410 Theory of Numbers, first mid-semester examination, February 14, 2011 ♡

Prof. Erich Kaltofen <kaltofen@math.ncsu.edu>
www.math.ncsu.edu/~kaltofen/courses/NumberTheory/Spring11/ (URL)

919.515.8785 (phone)
919.515.3798 (fax)

Your Name: ____________________________

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 8 questions, where each question counts for the explicitly given number of points, adding to a total of **44 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **one** 8.5in $\times$ 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **75 minutes** to do this test.

Good luck!

Problem 1	_____
2	_____
3	_____
4	_____
5	_____
Total	_____

Problem 1 (18 points)

(a, 5pts) Please give the solution (with an integer parameter λ) for the diophantine equation $138x + 384y = 18$ in the integer variables x and y. Please show your work.

(b, 4pts) Please list the first 4 Mersenne prime numbers.

(c, 5pts) Consider the factorization into primes of 100!, namely $100! = 2^{97}\, 3^{48}\, 5^{24} \cdots$.[1] How many distinct primes occur in the full factorization, and which primes occur once, i.e., have exponent 1? Please explain.

(d, 4pts) Please factor 29 in the Gaussian integers $\mathbb{G} = \{a + \boldsymbol{i}\, b \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$, where $\boldsymbol{i} = \sqrt{-1}$.

[1] By $n!$ we denote the factorial, $n! = 1 \cdot 2 \cdots (n-1) \cdot n$.

Problem 2 (8 points): Please prove for all integers $n \geq 0$:

$$\sum_{i=0}^{n} 2^{n-i}\binom{n}{n-i} = 2^n\binom{n}{n} + 2^{n-1}\binom{n}{n-1} + \cdots + 2\binom{n}{1} + \binom{n}{0} = 3^n.$$

Problem 3 (8 points): Consider the sequence a_n of rational numbers that is inductively defined for all integers $n \geq 0$ by $a_0 = 0$, $a_1 = 1/2$ and $a_{n+2} = a_{n+1} - \frac{1}{4}a_n$. Thus the next elements are $a_2 = 1/2$, $a_3 = 3/8$, $a_4 = 1/4, \ldots$ Please prove by induction that $a_n = n2^{-n}$ for all integers $n \geq 0$.

Problem 4 (5 points): True or false: for all integers $n \geq 2$ and all integers i with $2 \leq i \leq n$, none of the integers $n! + i$ are prime numbers. Please explain.

Problem 5 (5 points): True or false: for all integers $n \geq 2$ and all integers i with $2 \leq i \leq n!$, none of the integers $n! + i$ are prime numbers. Please explain. [Hint: Chebyshev.]