2009

Problem 1 (18 points)

(a, 5pts) Please compute $g = \gcd(224, 156)$ and $s, t \in \mathbb{Z}$ such that $224s + 156t = g$. Please show all work.

	q	s	t
224		1	0
156		0	1
68	1	1	−1
20	2	−2	3
8	3	7	−10
4	2	−16	23
0	2	39	−56

g 2pts

s, t 3pts

$4 = (-16)224 + 23 \cdot 156$

(b, 4pts) Please give a parametric solution for the diophantine equation $224x + 156y = 8$.

$224/4 = 56 \qquad 156/4 = 39$

$8 = (-32) \cdot 224 + 46 \cdot 156$

$x = -32 + \lambda 39$

$y = 46 - \lambda 56$

−32
46 +2pts

7
−10
+2pts

(c, 4pts) Please compute $\pi(105)$. Please show your work.

$\pi(100) = 25$, 101 and 103 are prime

So $\pi(105) = 27$

(d, 5pts) Alice and Bob are choosing from 6 objects, A, B, C, D, E, and F. First, Alice chooses 2 and then Bob chooses 3 from the rest. How many different results are possible?

$$\frac{6 \cdot 5}{1 \cdot 2} \cdot \frac{4 \cdot \cancel{3} \cdot \cancel{2}}{1 \cdot \cancel{2} \cdot \cancel{3}} = 15 \cdot 4 = 60$$

$\binom{6}{2} \cdot \binom{6}{3}$ +2 pts

$$\binom{6}{2} \cdot \binom{4}{3}$$

$\binom{6}{2} + \binom{4}{3}$ +2pts

false listing +1

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Problem 2 (8 points): Please prove for all integers $n \geq 1$: $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$.

$$\frac{1}{n+1} \frac{(2n)!}{n!\,n!} \qquad\qquad \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$\frac{(2n)!}{n!\,(n+1)!} \qquad\qquad \frac{1}{n}\frac{(2n)!}{(n-1)!\,n!} - \frac{1}{n+1}\frac{(2n)!}{(n-1)!\,n!}$$

$$= \qquad \parallel$$

$$\underbrace{\left(\frac{1}{n} - \frac{1}{n+1}\right)}_{\frac{1}{n(n+1)}} \frac{(2n)!}{(n-1)!\,n!}$$

Problem 3 (8 points): Consider the sequence a_n that is inductively defined for all integers $n \geq 0$ by $a_0 = 0$, $a_1 = 2$ and $a_{n+2} = 4a_{n+1} - 4a_n$. Thus the next elements are $a_2 = 8$, $a_3 = 24$, $a_4 = 64, \ldots$ Please prove that $a_n = n \cdot 2^n$ for all integers $n \geq 0$.

$n=0 \qquad a_0 = 0 = 0 \cdot 2^0$

$n=1 \qquad a_1 = 2 = 1 \cdot 2^1$

+2

Hypo: $\forall\, 0 \leq n \leq k: \; a_n = n \cdot 2^n$

Ind. arg: $k+1$:

$$a_{k+1} = 4a_k - 4a_{k-1} \quad \text{(recursion)}$$

$$= 4k \cdot 2^k - 4(k-1)2^{k-1}$$

$$= (2k - (k-1))\, 2^{k+1} \qquad +6$$

$$= (k+1)\, 2^{k+1}$$

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Problem 4 (5 points): Please find all twin prime pairs (F_n, M_p), where F_n is a Fermat prime and M_p is a Mersenne prime.

$$(q, q+2) = (\underbrace{2^{2^n}+1}_{q}, \underbrace{2^p-1}_{q+2})$$

$$2^{2^n}+3 = 2^p - 1 \qquad 4(2^{2^n-2}+1) = 2^p$$

Hence $n = 1$, $2^{2^n-2}+1 = 2^0+1 = 2$

$p = 3$

$$(2^{2^1}+1 = 5, \; 2^3-1 = 7)$$

$n = 0$: $2^{2^0}+1 = 3$, $3+2 = 5 \neq 2^p - 1$

$F_n = M_p$ +2 pts

(3, 5)

~~ex~~

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Problem 5 (5 points): Please state the Green-Tao theorem on primes in an arithmetic progression.

$\forall n \geq 1 \; \exists a, b$: $a + kb$ prime for all $0 \leq k \leq n$

$\forall n \geq 1 \; \exists a, b \in \mathbb{Z} \; \forall k, 0 \leq k \leq n$: $a + kb$ prime

(arbitrarily long sequences of equidistant primes)