2007

Problem 1 (18 points)

(a, 5pts) Please compute $g = \gcd(325, 120)$ and $s, t \in \mathbb{Z}$ such that $\overset{325}{\cancel{272}}s + \overset{120}{\cancel{119}}t = g$. Please show all work.

q			
	325	1	0
q	120	0	1
2	85	1	−2
1	35	−1	3
2	15	3	−8
2	5	−7	19
3	0		

$(-7)\,325 + 19 \cdot 120 = 5$

(b, 5pts) Please give the prime factorization of $\binom{12}{6}$.

$$\frac{\overset{2}{\cancel{12}} \cdot 11 \cdot \overset{\cancel{5}}{\cancel{10}} \cdot \overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot 7}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}} = 2^2 \cdot 3 \cdot 7 \cdot 11$$

(c, 4pts) Knowing that $p_{50} = 229$, (the 50-th prime number) please compute $\pi(232)$.

230, 231, 232

$= 3 \cdot 77$

$\pi(232) = 50$

(d, 4pts) You are choosing 10 times from 2 objects, A and B. How many combinations with repetition are possible?

$$\binom{n+i-1}{i-1} = \binom{10+1}{1} = 11$$

$$\left.\begin{array}{l} A \ldots A \\ A \ldots A\,B \\ A \ldots B\,B \\ \\ B \ldots B \end{array}\right\} 11$$

Problem 2 (8 points): Please prove for all integers $n \geq 1$ that $\sum_{i=1}^{n}(2i-1) = n^2$.

Basis: $n=1$: $(2\cdot 1-1) = 1^2$.

Hypothesis $\forall n,\ 1 \leq n \leq k$: $\sum_{i=1}^{n}(2i-1) = n^2$

Inductive argument : $n = k+1$

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + 2k+1$$

$$\underset{\text{hypo}}{=} k^2 + 2k+1 = (k+1)^2.$$

Problem 3 (8 points): Please prove by induction on n that for all integers $n \geq 1$, $3^{3n+1} + 2^{n+1}$ is divisible by 5.

Basis: $n=1$ — $3^4 + 2^2 = 81 + 4 = 85 = 5 \cdot 17$

Hypothesis: $\forall n,\ 1 \leq n \leq k\ \exists f(n) \in \mathbb{Z} : 3^{3n+1} + 2^{n+1} = 5 \cdot f(n)$

Inductive argument : $n = k+1$

$$3^{3(k+1)+1} + 2^{(k+1)+1} = 27 \cdot 3^{3k+1} + 2 \cdot 2^{k+1}$$

$$= 27\left(3^{3k+1} + 2^{k+1}\right) - 25\ 2^{k+1}$$

$$\underset{\text{hypo}}{=} 27 \cdot 5 \cdot f(k) - 25\ 2^{k+1}$$

$$= 5\left(27\ f(k) - 5\ 2^{k+1}\right) = 5 \cdot f(k+1)$$

2007

Problem 4 (5 points): Please list the first 3 Mersenne primes and the first 3 Fermat primes.

$M_2 = 2^2 - 1 = 3$ $\quad F_0 = 2^{2^0} + 1 = 3$

$M_3 = 2^3 - 1 = 7$ $\quad F_1 = 2^{2^1} + 1 = 5$

$M_5 = 2^5 - 1 = 31$ $\quad F_2 = 2^{2^2} + 1 = 17$

Problem 5 (5 points): Please state the prime number theorem.

$$\lim_{n \to \infty} \frac{\pi(n)}{n/\log_e n} = 1$$