

2007

Problem 1 (18 points)

(a, 5pts) Please compute $g = \gcd(325, 120)$ and $s, t \in \mathbb{Z}$ such that $325s + 120t = g$. Please show all work.

	325	1	0
2	120	0	1
2	85	1	-2
1	35	-1	3
2	15	3	-8
2	5	-7	19
3	0		

$$(-7) \cdot 325 + 19 \cdot 120 = 5$$

(b, 5pts) Please give the prime factorization of $\binom{12}{6}$.

$$\frac{\cancel{2} \cdot \cancel{11} \cdot \cancel{10} \cdot \cancel{9} \cdot \cancel{8} \cdot 7}{1 \cdot \cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3}} = 2^2 \cdot 3 \cdot 7 \cdot 11$$

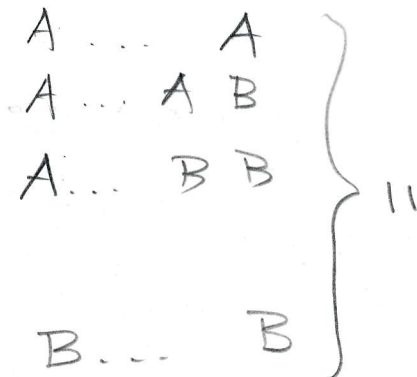
(c, 4pts) Knowing that $p_{50} = 229$, (the 50-th prime number) please compute $\pi(232)$.

$$230, 231, 232 \\ = 3 \cdot 77$$

$$\pi(232) = 50$$

(d, 4pts) You are choosing 10 times from 2 objects, A and B. How many combinations with repetition are possible?

$$\binom{n+i-1}{i-1} = \binom{10+1}{1} = 11$$



Problem 2 (8 points): Please prove for all integers $n \geq 1$ that $\sum_{i=1}^n (2i-1) = n^2$.

Basis: $n=1$: $(2 \cdot 1 - 1) = 1^2$

Hypothesis $\forall n, 1 \leq n \leq k$: $\sum_{i=1}^n (2i-1) = n^2$

Inductive argument: $n=k+1$

$$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^k (2i-1) + 2k+1$$

$$= k^2 + 2k + 1 = (k+1)^2$$

hypo

Problem 3 (8 points): Please prove by induction on n that for all integers $n \geq 1$, $3^{3n+1} + 2^{n+1}$ is divisible by 5.

Basis: $n=1$: $3^4 + 2^2 = 81 + 4 = 85 = 5 \cdot 17$

Hypothesis: $\forall n, 1 \leq n \leq k \exists f(n) \in \mathbb{Z}: 3^{3n+1} + 2^{n+1} = 5 \cdot f(n)$

Inductive argument: $n=k+1$

$$3^{3(k+1)+1} + 2^{(k+1)+1} = 27 \cdot 3^{3k+1} + 2 \cdot 2^{k+1}$$

$$= 27(3^{3k+1} + 2^{k+1}) - 25 \cdot 2^{k+1}$$

$$\stackrel{\text{hypo}}{=} 27 \cdot 5 \cdot f(k) - 25 \cdot 2^{k+1}$$

$$= 5(27 f(k) - 5 \cdot 2^{k+1}) = 5 \cdot f(k+1)$$

2007

Problem 4 (5 points): Please list the first 3 Mersenne primes and the first 3 Fermat primes.

$$M_2 = 2^2 - 1 = 3$$

$$F_0 = 2^{2^0} + 1 = 3$$

$$M_3 = 2^3 - 1 = 7$$

$$F_1 = 2^{2^1} + 1 = 5$$

$$M_5 = 2^5 - 1 = 31$$

$$F_2 = 2^{2^2} + 1 = 17$$

Problem 5 (5 points): Please state the prime number theorem.

$$\lim_{n \rightarrow \infty} \frac{\pi(n)}{n / \log n} = 1$$