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Department of Mathematics

NC STATE UNIVERSITY

MA 410 Theory of Numbers, first mid-semester examination, Feb 14, 2005

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Your Name: SOLUTION
For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 9 questions, where each question counts for the explicitly given number of points, adding to a total of **44 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work,** if necessary. You are allowed to consult **one** 8.5in $\times$ 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **60 minutes** to do this test.

Good luck!

Problem 1	____
2	____
3	____
4	____
5	____
Total	____

Problem 1 (18 points)

(a, 5pts) Please compute $g = \gcd(272, 119)$ and $s,t \in \mathbb{Z}$ such that $272s + 119t = g$. Please show all work.

$$\begin{array}{rcc} & 1 & 0 \\ & 0 & 1 \\ 272 - 2 \cdot 119 = 34 & 1 & -2 \\ 119 - 3 \cdot 34 = 17 & -3 & 7 \\ 34 - 2 \cdot 17 = 0 & & \end{array}$$

$$(-3) \cdot 272 + 7 \cdot 119 = 17 = \text{GCD}(272, 119).$$

(b, 4pts) Please compute the binomial coefficient $\binom{9}{4}$. Please show all work.

$\frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 9 \cdot 7 \cdot 2 = 126.$

(c, 5pts) Please compute p_{20} (the 20-th prime number) and $\pi(50)$.

$p_{20} = 71, \pi(50) = 15.$

(d, 4pts) Please give the prime factorization of 14!.

$14! = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13.$

Problem 2 (8 points): Please prove that $\gcd(17a+12, 3a+2) = 2$ for all even integers a, and that $\gcd(17a+12, 3a+2) = 1$ for all odd integers a.

$3(17a+12) - 17(3a+2) = 36 - 34 = 2$, *hence GCD*$(17a+12, 3a+2)$ *divides* 2.
If a is even, $17a+12$ *and* $3a+2$ *are even, hence their GCD is 2.*
If a is odd, $17a+12$ *is odd, hence their GCD is 1.*

Problem 3 (8 points): Consider the sequence a_n that is inductively defined for all integers $n \geq 0$ by $a_0 = 2$, $a_1 = 1$ and $a_{n+2} = a_{n+1} + 2 \cdot a_n$. Thus the next elements are $a_2 = 5$, $a_3 = 7$, $a_4 = 17$, $a_5 = 31$, $a_6 = 65, \ldots$ Please prove that $a_n = 2^n + (-1)^n$ for all integers $n \geq 0$.

Basis: $n = 0$*:* $a_0 = 2 = 2^0 + (-1)^0$. $n = 1$*:* $a_1 = 2^1 + (-1)^1 = 1$.
Hypothesis: for $0 \leq n \leq k$ *we have* $a_n = 2^n + (-1)^n$.
Induction argument for $n = k+1$*:*

$$\begin{aligned}
a_{k+1} &= a_k + 2a_{k-1} \\
&= 2^k + (-1)^k + 2 \cdot 2^{k-1} + 2 \cdot (-1)^{k-1} \\
&= 2^{k+1} + (-1)^{k-1}(-1+2) \\
&= 2^{k+1} + (-1)^{k+1}.
\end{aligned}$$

Problem 4 (5 points): True or false: in $\mathbb{Z}_{>0}$ there exist 2^{100} **consecutive** composite integers. Please explain.

TRUE: $(2^{100}+1)!+2, (2^{100}+1)!+3, \dots (2^{100}+1)!+(2^{100}+1)$ *are composite numbers.*

Problem 5 (5 points): True or false: Between 2^{100} and 2^{101} there is at least one prime number. Please explain.

TRUE: this is Chebychev's theorem.