NC STATE UNIVERSITY

MA 410 Theory of Numbers, first mid-semester examination, Feb 14, 2005 ${\tt kaltofen@math.ncsu.edu}$ (email)

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Your Name: SOLUTION

For purpose of anonymous grading, please do **not** write your name on the subsequent pages.

This examination consists of 5 problems, which are subdivided into 9 questions, where each question counts for the explicitly given number of points, adding to a total of **44 points**. Please write your answers in the spaces indicated, or below the questions, using the **back of the sheets** for completing the answers and **for all scratch work**, if necessary. You are allowed to consult **one** 8.5in \times 11in sheet with notes, but **not** your book or your class notes. If you get stuck on a problem, it may be advisable to go to another problem and come back to that one later.

You will have **60 minutes** to do this test.

Good luck!

Problem 1	
2	
3	
4	
5	
Total	

Problem 1 (18 points)

(a, 5pts) Please compute $g = \gcd(272, 119)$ and $s, t \in \mathbb{Z}$ such that 272s + 119t = g. Please show all work.

$$\begin{array}{ccccc}
 & 1 & 0 \\
 & 0 & 1 \\
272 - 2 \cdot 119 = 34 & 1 & -2 \\
119 - 3 \cdot 34 = 17 & -3 & 7 \\
34 - 2 \cdot 17 = 0
\end{array}$$

$$(-3) \cdot 272 + 7 \cdot 119 = 17 = GCD(272, 119).$$

(b, 4pts) Please compute the binomial coefficient $\binom{9}{4}$. Please show all work.

$$\frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 9 \cdot 7 \cdot 2 = 126.$$

(c, 5pts) Please compute p_{20} (the 20-th prime number) and $\pi(50)$.

$$p_{20} = 71, \pi(50) = 15.$$

(d, 4pts) Please give the prime factorization of 14!.

$$14! = 2^{11} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13.$$

Problem 2 (8 points): Please prove that gcd(17a + 12, 3a + 2) = 2 for all even integers a, and that gcd(17a + 12, 3a + 2) = 1 for all odd integers a.

3(17a+12)-17(3a+2)=36-34=2, hence GCD(17a+12,3a+2) divides 2. If a is even, 17a+12 and 3a+2 are even, hence their GCD is 2. If a is odd, 17a+12 is odd, hence their GCD is 1.

Problem 3 (8 points): Consider the sequence a_n that is inductively defined for all integers $n \ge 0$ by $a_0 = 2$, $a_1 = 1$ and $a_{n+2} = a_{n+1} + 2 \cdot a_n$. Thus the next elements are $a_2 = 5$, $a_3 = 7$, $a_4 = 17$, $a_5 = 31$, $a_6 = 65$,... Please prove that $a_n = 2^n + (-1)^n$ for all integers $n \ge 0$.

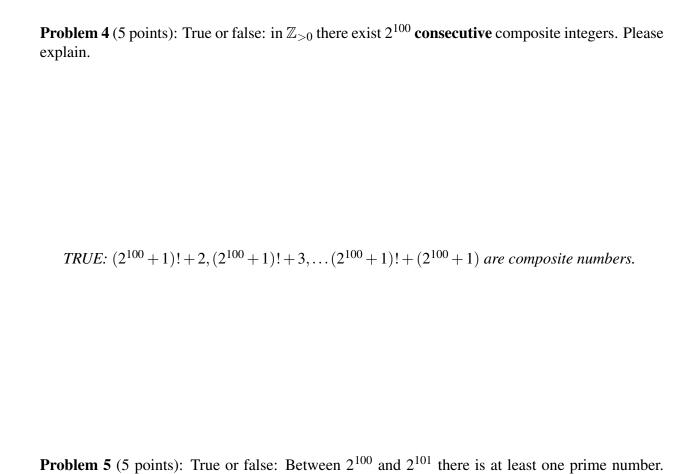
Basis: n = 0: $a_0 = 2 = 2^0 + (-1)^0$. n = 1: $a_1 = 2^1 + (-1)^1 = 1$. Hypothesis: for $0 \le n \le k$ we have $a_n = 2^n + (-1)^n$. Induction argument for n = k + 1:

$$a_{k+1} = a_k + 2a_{k-1}$$

$$= 2^k + (-1)^k + 2 \cdot 2^{k-1} + 2 \cdot (-1)^{k-1}$$

$$= 2^{k+1} + (-1)^{k-1} (-1+2)$$

$$= 2^{k+1} + (-1)^{k+1}.$$



Please explain.

TRUE: this is Chebychev's theorem.