Fair Division and Apportionment

1 Introduction

Fair Division Divide objects *equally* among players.

- Continuous or discrete.
- Objects may be different.
- Each player gets "fair share", not necessarily *best* share.
- Shares equal, objects different.
- E.g., dividing cake (continuous) or candy (discrete) among siblings.

Apportionment Divide objects *proportionally* among players.

- Only discrete.
- Objects are equal.
- Each player gets different amount.
- E.g., assigning congressional seats to US House of Representatives, or state congresses.

2 Fair Division

2.1 Definitions

Set of goods Things to be divided. S.

Set of players Parties entitled to share set of goods. $P_1, \ldots, P_N$.

Fair share Any share that *in the opinion of the player receiving it* is worth *at least one* N*-th* of the total value.

- It only matters what the one player thinks.
- What other players think is irrelevant.

2.2 Division Schemes

Continuous Set divisible infinitely many ways. (Cake, land, time, ...)

Discrete Set of goods composed of objects that are indivisible. (Paintings, houses, cars, jewelry, candy, ...)

Mixed Some components continuous and some mixed.

2.3 Two Players: The Divider-Chooser Method

- Continuous method.
  1. *Divider* divides object into two equal pieces.
  2. *Chooser* chooses piece they feel to be bigger.
- All opinions are secret before hand.
- Divider at disadvantage; should be picked randomly.

2.4 Lone Divider Method

- Continuous method.
  1. Divider divides cake into three equal objects.
  2. Each chooser declares (secretly) which pieces are fair shares.
  3. Divide by bids:
     (a) If possible, give each player a piece they bid for. Divider gets last piece
     (b) If only one piece bid one, combine it with one of other pieces, and choosers divide large piece. Divider gets remaining piece.
- All opinions are secret before hand.
- Divider at disadvantage; should be picked randomly.

2.5 Lone Chooser Method

- Continuous method.
  1. Dividers cut cake, among themselves, into two fair shares.
  2. Each divider divides his share into three equal shares.
  3. Chooser chooses one piece from each divider.
- All opinions are secret before hand.

2.6 The Last Diminisher Method

- Continuous method.

  1. Players randomly assigned order.
  2. First player cut slice to be exact fair share.
  3. For each of remaining players, if player believes piece is more than fair share, they claim it, and cut off piece so remaining part if exact fair share. Otherwise player passes.
  4. Last player to claim piece receives it.
  5. Repeat process after removing player with piece.

- All opinions are secret before hand.

2.7 The Method of Sealed Bids

- Discrete method.

  1. Each player gives *honest* (and secret or *sealed*) bid for each item in estate.
  2. Each item goes to highest bidder.
  3. Each player *pays* or is *paid by* the estate to make fair share.
     - For each player, figure their fair share based on their bids.
     - Each player pays or receives money to account for their fair share.
     - Estate will make money!
  4. divide surplus equally among players.

- All opinions are secret before hand.
- Each player both buyer and seller.
- Each player must have enough money to play the game.
- Each player must be willing to accept money instead of objects.

2.8 Method of Markers

- Discrete method.

  1. Line objects in array.
  2. Each player divides array into N acceptable segments by placing $N - 1$ markers.
  3. Scan array from left to right for first marker. Player owning marker gets first segment and all of his/her markers removed. Break ties randomly.

4. Continue scanning from left to right for first *second* marker. Player owning it gets their second section.
5. Continue process for entire array.
6. Divide left overs by lottery.

- All opinions are secret before hand.
- Each player gets more than fair share with lottery.

3 Apportionment

3.1 Definitions

Standard divisor Number of people per seat in legislature.

$$\text{Standard divisor} = \frac{\text{population}}{\text{total number of seats}}$$

Standard Quota Fraction of total state entitled to if seats were not indivisible.

$$\text{State's standard quota} = \frac{\text{state's population}}{\text{standard divisor}}$$

Lower Quota Standard quota rounded down.

Upper Quota Standard quota rounded up.

Quota Rule A state's apportionment should either be it's upper or lower quota.

3.2 Hamilton's Method

- Mathematically the simplest.
- Proposed by Alexander Hamilton.
- Approved by Congress in 1791; vetoed by President Washington. (First use of presidential veto.)
- Approved by Congress in 1852 until 1901.

1. Calculate each state's standard quota.
2. Give each state lower quota. (Round down.)
3. Give surplus seats one at a time to states with largest fractional parts until out of surplus seats.

- Satisfies quota rule.

- Most serious flaw is *Alabama paradox*: Increase in total number of seats can in and of itself force a state to lose seats.
  - In 1880, Alabama would get 8 seats from total of 299, but only 7 from total of 300.
  - Reason is it changes priority order of assigning surplus seats.
- *Population Paradox*: State X could lose seats to state Y even though population of X had grown faster than population of Y.
- *New States Paradox*: Adding new state, and increasing number of seats, can cause another state to loose seats.
  - In 1907, Oklahoma added as new state with 5 new seats to house (386 to 391). Maine's apportionment went up (3 to 4), and New York's went down (38 to 37).
  - Reason is it changes priority order of assigning surplus seats.

3.3 Jefferson's Method

- Proposed by Thomas Jefferson.
- Approved by Congress in 1791 after veto of Hamilton's Method. Used until 1842.
- Do not use standard divisor, but a different divisor and *modified quota* such that rounding *down* gives no surplus seats.

1. Find divisor D so rounding modified quota downward gives no surplus seats.
2. Apportion seats by lower quota.

- Does not satisfy quota rule. Upper quota violation.

3.4 Adam's Method

- Proposed by John Quincy Adams.
- Do not use standard divisor, but a different divisor and *modified quota* such that rounding *up* gives no surplus seats.

1. Find divisor D so rounding modified quota upward gives no surplus seats.
2. Apportion seats by lower quota.

- Does not satisfy quota rule. Lower quota violation.

3.5 Webster's Method

- Proposed by Daniel Webster.
- Used 1842-1852 and 1901-1941.
- Do not use standard divisor, but a different divisor and *modified quota* such that rounding conventionally gives no surplus seats. Cut-off point for rounding is algebraic mean of lower and upper quotas.

$$H = \frac{L + (L + 1)}{2}$$

1. Find divisor D so rounding modified quota gives no surplus seats.
2. Apportion seats by rounded quota.

- Does not satisfy quota rule.

3.6 Huntington-Hill Method

- Proposed around 1911 by Joseph A. Hill, chief statistician at the Bureau of Census, and Edward V. Huntington, professor of mechanics and mathematics at Harvard University.
- Used 1941 to present.
- Do not use standard divisor, but a different divisor and *modified quota* such that rounding with cut-off point at geometric mean

$$H = \sqrt{L(L + 1)}$$

gives no surplus seats.

1. Find divisor D so rounding modified quota at geometric mean gives no surplus seats.
2. Apportion seats by rounded quota.

- Does not satisfy quota rule.

3.7 Balinsky and Young's Impossibility Theorem

Theorem 1 (Baliknsy-Young Impossibility Theorem). *There are no perfect apportionment methods. Any apportionment method that does not violate the quota rule must produce paradoxes, and any apportionment method that does not produce paradoxes must violate the quota rule.*