```
> restart:
> f := factor(x^6 - 2*x^5 - 2*x^4 + 2*x^2 - 2*x - 1);
```

$$f := x^6 - 2x^5 - 2x^4 + 2x^2 - 2x - 1 \tag{1}$$

```
> alias(y1 = RootOf(f,x));
```

$$y1 \tag{2}$$

```
> facs_y1:=evala(Factor(f,y1));
```

$$facs_y1 := \left(-\frac{y1^5}{2} + \frac{3y1^4}{2} - \frac{y1^3}{2} + \frac{y1^2}{2} - \frac{y1}{2} + \frac{1}{2} + (y1-1)x + x^2\right)\left(-\frac{y1^5}{2} + \frac{3y1^4}{2} + \frac{y1^3}{2} - \frac{3y1^2}{2} - \frac{5y1}{2} + \frac{1}{2} + \left(-y1^5 + 2y1^4 + 2y1^3 - 2y1 + 1\right)x + x^2\right)\left(y1^5 - 2y1^4 - 2y1^3 + x + 2y1 - 2\right)(x - y1) \tag{3}$$

```
> f1 := op(1,facs_y1);
```

$$f1 := -\frac{y1^5}{2} + \frac{3y1^4}{2} - \frac{y1^3}{2} + \frac{y1^2}{2} - \frac{y1}{2} + \frac{1}{2} + (y1-1)x + x^2 \tag{4}$$

```
> alias(y2 = RootOf(f1,x));
```

$$y1, y2 \tag{5}$$

```
> facs_y1_y2 := evala(Factor(f,{y2,y1}));
```

$$facs_y1_y2 := (x - y2)(x + y2 + y1 - 1)\left(x - y1^5 + 2y1^4 + 2y1^3 - 2y1 + 1 + \frac{1}{2}y1^5y2 - \frac{3}{2}y1^4y2 - \frac{1}{2}y1^3y2 + \frac{3}{2}y1^2y2 + \frac{5}{2}y2y1 - \frac{1}{2}y2\right)\left(-\frac{1}{2}y1^5y2 + \frac{3}{2}y1^4y2 + \frac{1}{2}y1^3y2 - \frac{3}{2}y1^2y2 - \frac{5}{2}y2y1 + x + \frac{1}{2}y2\right)\left(y1^5 - 2y1^4 - 2y1^3 + x + 2y1 - 2\right)(x - y1) \tag{6}$$

```
> facs_z := subs({y1=z1,y2=z2},facs_y1_y2);
```

$$facs_z := (x - z2)(x + z2 + z1 - 1)\left(x - z1^5 + 2z1^4 + 2z1^3 - 2z1 + 1 + \frac{1}{2}z1^5z2 - \frac{3}{2}z1^4z2 - \frac{1}{2}z1^3z2 + \frac{3}{2}z1^2z2 + \frac{5}{2}z2z1 - \frac{1}{2}z2\right)\left(-\frac{1}{2}z1^5z2 + \frac{3}{2}z1^4z2 + \frac{1}{2}z1^3z2 - \frac{3}{2}z1^2z2 - \frac{5}{2}z2z1 + x + \frac{1}{2}z2\right)\left(z1^5 - 2z1^4 - 2z1^3 + x + 2z1 - 2\right)(x - z1) \tag{7}$$

```
> f1_z := subs({y1=z1,x=z2},f1);f_z := subs(x=z1,f);
```

$$f1_z := -\frac{z1^5}{2} + \frac{3z1^4}{2} - \frac{z1^3}{2} + \frac{z1^2}{2} - \frac{z1}{2} + \frac{1}{2} + (z1-1)z2 + z2^2$$

$$f_z := z1^6 - 2z1^5 - 2z1^4 + 2z1^2 - 2z1 - 1 \tag{8}$$

```
> F := expand(facs_z);
```

$$F := \frac{91}{4}z1^5z2^3x^2 - \frac{43}{4}z1^2z2^3x^2 + \frac{5}{2}x^4z1z2^2 - \frac{5}{2}x^2z1z2^4 - \frac{9}{2}x^4z1z2 - \frac{7}{2}x^4z1^4z2 + \frac{7}{2}x^4z1^2z2 + \frac{9}{2}x^4z1^3z2 + \frac{41}{4}z1^4z2^3x^2 - \frac{31}{4}z1^3z2^3x^2 + \frac{25}{4}z1z2^3x^2 + \frac{3}{2}x^4z2 + 11x^4z1^5 - 19x^2z1^7 - \frac{1}{2}xz2^4 + \frac{1}{4}x^2z2^4 + 2x^4z1^4 - 6x^4z1^3 - \frac{1}{4}z1^{17}z2^3 \tag{9}$$

$$-\frac{1}{4}\,z1^{16}\,z2^{4}+\frac{1}{2}\,z1^{17}\,z2^{2}-x^{4}\,z1^{10}+x^{2}\,z1^{12}+4\,x^{4}\,z1^{9}-8\,x^{4}\,z1^{7}-8\,x^{4}\,z1^{6}-5\,x^{4}\,z1^{2}$$
$$-\frac{5}{4}\,x^{4}\,z2^{2}-\frac{3}{4}\,z2^{3}\,x^{2}-2\,x^{3}\,z1^{4}+6\,x^{3}\,z1^{3}+\frac{11}{4}\,z1^{16}\,z2^{3}+2\,z1^{15}\,z2^{4}-4\,z1^{16}\,z2^{2}$$
$$-\frac{39}{4}\,z1^{15}\,z2^{3}-\frac{17}{4}\,z1^{14}\,z2^{4}+9\,z1^{15}\,z2^{2}+\frac{29}{4}\,z1^{14}\,z2^{3}-\frac{5}{2}\,z1^{13}\,z2^{4}+2\,z1^{14}\,z2^{2}$$
$$+\frac{83}{4}\,z1^{13}\,z2^{3}+\frac{43}{4}\,z1^{12}\,z2^{4}-19\,z1^{13}\,z2^{2}-\frac{27}{4}\,z1^{12}\,z2^{3}+\frac{31}{2}\,z1^{11}\,z2^{4}-\frac{41}{2}\,z1^{12}\,z2^{2}$$
$$+x^{3}\,z1^{10}-5\,x^{2}\,z1^{11}-4\,x^{3}\,z1^{9}+4\,x^{2}\,z1^{10}+8\,x^{2}\,z1^{9}+8\,x^{3}\,z1^{7}+8\,x^{3}\,z1^{6}+7\,x^{4}\,z1$$
$$+5\,x^{3}\,z1^{2}+\frac{7}{4}\,x^{3}\,z2^{2}+z1\,z2^{2}+\frac{9}{2}\,z1^{6}\,z2^{2}+47\,z1^{5}\,z2^{3}+\frac{9}{2}\,z1^{4}\,z2^{3}-34\,z1^{3}\,z2^{3}$$
$$+14\,z1^{2}\,z2^{3}-9\,z1^{2}\,z2^{2}-\frac{3}{2}\,z1\,z2^{3}-x^{2}\,z2-18\,z2^{4}\,z1^{5}+7\,z2^{4}\,z1^{4}+\frac{29}{2}\,z2^{4}\,z1^{3}$$
$$-\frac{11}{2}\,z2^{4}\,z1^{2}+\frac{1}{2}\,z2^{4}\,z1+\frac{3}{2}\,z2^{3}\,x+8\,x^{2}\,z1^{2}+\frac{67}{4}\,z1^{6}\,z2^{3}-\frac{311}{4}\,z1^{11}\,z2^{3}$$
$$+\frac{115}{4}\,z1^{10}\,z2^{3}+\frac{375}{4}\,z1^{9}\,z2^{3}+\frac{55}{4}\,z1^{8}\,z2^{3}-\frac{429}{4}\,z1^{7}\,z2^{3}-\frac{121}{4}\,z1^{10}\,z2^{4}-27\,z1^{9}\,z2^{4}$$
$$+\frac{93}{4}\,z1^{8}\,z2^{4}+\frac{87}{2}\,z1^{7}\,z2^{4}-\frac{69}{4}\,z1^{6}\,z2^{4}-\frac{5}{2}\,x^{3}\,z2+2\,x^{2}\,z2^{2}-7\,x^{3}\,z1-11\,x^{3}\,z1^{5}$$
$$+9\,x^{2}\,z1^{6}+\frac{115}{2}\,z1^{11}\,z2^{2}+18\,z1^{10}\,z2^{2}-50\,z1^{9}\,z2^{2}-40\,z1^{8}\,z2^{2}+64\,z1^{7}\,z2^{2}-2\,x^{2}\,z1$$
$$-x\,z2^{2}+8\,x^{2}\,z1^{5}-25\,z1^{5}\,z2^{2}-2\,x^{2}\,z1^{4}-11\,z1^{4}\,z2^{2}-10\,x^{2}\,z1^{3}+22\,z1^{3}\,z2^{2}$$
$$-\frac{1}{4}\,x^{4}\,z1^{10}\,z2^{2}+\frac{43}{4}\,x^{3}\,z1^{11}\,z2^{2}+\frac{15}{2}\,x^{3}\,z1^{12}\,z2+\frac{1}{4}\,x^{2}\,z1^{11}\,z2^{3}+\frac{1}{4}\,x^{2}\,z1^{10}\,z2^{4}$$
$$+\frac{1}{2}\,x^{4}\,z1^{10}\,z2-\frac{23}{2}\,x^{3}\,z1^{11}\,z2+\frac{3}{2}\,x^{4}\,z1^{9}\,z2^{2}-\frac{9}{4}\,x^{2}\,z1^{10}\,z2^{3}-\frac{3}{2}\,x^{2}\,z1^{9}\,z2^{4}$$
$$-\frac{5}{2}\,x^{4}\,z1^{9}\,z2-\frac{7}{4}\,x^{4}\,z1^{8}\,z2^{2}+\frac{23}{4}\,x^{2}\,z1^{9}\,z2^{3}+\frac{7}{4}\,x^{2}\,z1^{8}\,z2^{4}+\frac{3}{2}\,x^{4}\,z1^{8}\,z2-3\,x^{4}\,z1^{7}\,z2^{2}$$
$$-\frac{1}{4}\,x^{2}\,z1^{8}\,z2^{3}+3\,x^{2}\,z1^{7}\,z2^{4}+\frac{11}{2}\,x^{4}\,z1^{7}\,z2+\frac{7}{4}\,x^{4}\,z1^{6}\,z2^{2}-\frac{41}{4}\,x^{2}\,z1^{7}\,z2^{3}$$
$$-\frac{7}{4}\,x^{2}\,z1^{6}\,z2^{4}+\frac{3}{2}\,x^{4}\,z1^{6}\,z2+\frac{19}{2}\,x^{4}\,z1^{5}\,z2^{2}-\frac{37}{4}\,x^{2}\,z1^{6}\,z2^{3}-\frac{19}{2}\,x^{2}\,z1^{5}\,z2^{4}$$
$$-\frac{5}{4}\,x^{4}\,z1^{4}\,z2^{2}+\frac{5}{4}\,x^{2}\,z1^{4}\,z2^{4}-8\,x^{4}\,z1^{3}\,z2^{2}+8\,x^{2}\,z1^{3}\,z2^{4}-\frac{19}{4}\,x^{4}\,z1^{2}\,z2^{2}+\frac{19}{4}\,x^{2}\,z1^{2}\,z2^{4}$$
$$-\frac{29}{4}\,z1^{13}\,z2^{3}\,x+\frac{5}{2}\,z1^{12}\,z2^{4}\,x-21\,z1^{12}\,z2^{3}\,x-11\,z1^{11}\,z2^{4}\,x-12\,x^{4}\,z1^{5}\,z2$$
$$-\frac{1}{4}\,x^{3}\,z1^{15}\,z2^{2}+\frac{1}{4}\,x\,z1^{17}\,z2^{2}+\frac{1}{2}\,x^{3}\,z1^{15}\,z2+2\,x^{3}\,z1^{14}\,z2^{2}-\frac{1}{2}\,x\,z1^{17}\,z2$$
$$-\frac{7}{2}\,x^{3}\,z1^{14}\,z2-\frac{17}{4}\,x^{3}\,z1^{13}\,z2^{2}+\frac{11}{2}\,x^{3}\,z1^{13}\,z2-\frac{5}{2}\,x^{3}\,z1^{12}\,z2^{2}+\frac{59}{2}\,x^{3}\,z1^{5}\,z2$$
$$+\frac{1}{4}\,x^{2}\,z1^{15}\,z2^{2}-\frac{11}{4}\,x\,z1^{16}\,z2^{2}-\frac{1}{2}\,x^{2}\,z1^{15}\,z2-2\,x^{2}\,z1^{14}\,z2^{2}+4\,x\,z1^{16}\,z2$$
$$+\frac{41}{4}\,x\,z1^{15}\,z2^{2}+\frac{7}{2}\,x^{2}\,z1^{14}\,z2+\frac{17}{4}\,x^{2}\,z1^{13}\,z2^{2}-9\,x\,z1^{15}\,z2-\frac{43}{4}\,x\,z1^{14}\,z2^{2}$$
$$-\frac{11}{2}\,x^{2}\,z1^{13}\,z2+\frac{11}{4}\,x^{2}\,z1^{12}\,z2^{2}-2\,x\,z1^{14}\,z2-\frac{61}{4}\,x\,z1^{13}\,z2^{2}+\frac{63}{4}\,x^{3}\,z1^{10}\,z2^{2}$$
$$-8\,x^{2}\,z1^{12}\,z2-13\,x^{2}\,z1^{11}\,z2^{2}+19\,x\,z1^{13}\,z2+\frac{59}{4}\,x\,z1^{12}\,z2^{2}+9\,x\,z1^{11}\,z2^{3}-14\,x\,z1^{10}\,z2^{4}$$

$$
-\frac{65}{2}x^3 z1^{10} z2 - \frac{127}{4}x^3 z1^9 z2^2 + \frac{31}{2}x^2 z1^{11} z2 + \frac{39}{2}x z1^{12} z2 + 72 x z1^{10} z2^3
+\frac{57}{2}x z1^9 z2^4 + 28 x^3 z1^9 z2 - \frac{101}{4}x^3 z1^8 z2^2 - \frac{57}{2}x z1^9 z2^3 + 24 x z1^8 z2^4
+42 x^3 z1^8 z2 + \frac{105}{4}x^3 z1^7 z2^2 - \frac{167}{2}x z1^8 z2^3 - \frac{43}{2}x z1^7 z2^4 - 12 x^3 z1^7 z2
+\frac{167}{4}x^3 z1^6 z2^2 - \frac{9}{2}x z1^7 z2^3 - 34 x z1^6 z2^4 - 48 x^3 z1^6 z2 - \frac{107}{4}x^3 z1^5 z2^2
+\frac{169}{2}x z1^6 z2^3 + 16 x z1^5 z2^4 - \frac{67}{4}x^3 z1^4 z2^2 + 10 x z1^4 z2^4 + 15 x^3 z1^3 z2^2
-\frac{47}{4}x z1^3 z2^4 + \frac{77}{4}x^3 z1^2 z2^2 - 12 x z1^2 z2^4 - 7 x^3 z1 z2^2 + \frac{21}{4}x z1 z2^4 + \frac{1}{4}z1^{16} z2^3 x
+\frac{1}{4}z1^{15} z2^4 x - \frac{11}{4}z1^{15} z2^3 x - 2 z1^{14} z2^4 x + \frac{39}{4}z1^{14} z2^3 x + \frac{17}{4}z1^{13} z2^4 x
-\frac{211}{4}x z1^{10} z2^2 - \frac{257}{4}x z1^9 z2^2 + \frac{71}{4}x z1^8 z2^2 + \frac{317}{4}x z1^7 z2^2 - \frac{45}{4}x^2 z1^6 z2^2
+\frac{23}{2}x^3 z1 z2 + \frac{47}{2}x^2 z1^9 z2^2 + \frac{73}{4}x^2 z1^8 z2^2 - 19 x^2 z1^7 z2^2 + \frac{51}{2}x^3 z1^4 z2
-\frac{33}{2}x^3 z1^2 z2 - \frac{15}{2}x^3 z1^3 z2 - \frac{33}{4}x^2 z1^{10} z2^2 - \frac{157}{4}z1^4 z2^3 x - 18 z1^2 z2^2 x
+\frac{9}{2}x^2 z2^2 z1^5 - \frac{51}{2}x^2 z1^9 z2 - \frac{27}{2}x z1^6 z2 - \frac{39}{2}x z2^2 z1^5 + 7 z1 z2^2 x - \frac{3}{4}x^2 z2^2 z1^2
-\frac{105}{2}x z1^{11} z2 + \frac{25}{4}z1^3 z2^3 x - \frac{29}{4}x^2 z2^2 z1^3 - 22 x z1^{10} z2 + 20 x^2 z1^7 z2
+\frac{3}{2}x^2 z1 z2 - 45 x z1^7 z2 + 23 x^2 z1^{10} z2 - \frac{63}{2}x^2 z1^8 z2 + 19 x^2 z1^6 z2 - \frac{53}{4}z1 z2^3 x
+13 x z1^4 z2 - 12 x z1^3 z2 - 13 x^2 z1^4 z2 + \frac{249}{4}x z1^{11} z2^2 - 27 z1^5 z2^3 x + 17 x z1^5 z2
-x^2 z1^2 z2 + x z1^2 z2 + 8 x^2 z1^3 z2 + \frac{19}{4}x^2 z2^2 z1^4 + x z1 z2 - \frac{23}{4}x^2 z2^2 z1 + 42 x z1^9 z2
-\frac{33}{2}x^2 z1^5 z2 + \frac{111}{4}z1^2 z2^3 x + \frac{21}{2}x z2^2 z1^4 + 40 x z1^8 z2 + 20 x z2^2 z1^3
-\frac{175}{4}z1^6 z2^2 x + 2 x^3 + x^6 - 2 x^5 - x^4
$$

```
> expand(rem(F,f1_z,z2));
```

$$
-\frac{3}{4}x^4 z1 z2 + \frac{13}{4}x^4 z1^4 z2 - \frac{15}{4}x^4 z1^2 z2 + \frac{5}{4}x^4 z1^3 z2 + \frac{1}{4}x^4 z2 + \frac{23}{8}x^4 z1^5 \qquad \textbf{(10)}
-\frac{73}{4}x^2 z1^7 + \frac{33}{8}x^4 z1^4 - \frac{25}{4}x^4 z1^3 + \frac{11}{8}x^4 z1^{10} - \frac{421}{16}x^2 z1^{12} - \frac{81}{8}x^4 z1^9
-\frac{11}{8}x^4 z1^7 + \frac{7}{8}x^4 z1^6 - \frac{3}{4}x^4 z1^2 - \frac{15}{8}x^3 z1^4 + \frac{25}{2}x^3 z1^3 - 73 x^3 z1^{10} + \frac{109}{8}x^2 z1^{11}
+\frac{155}{4}x^3 z1^9 + 67 x^2 z1^{10} - \frac{233}{8}x^2 z1^9 + \frac{63}{8}x^3 z1^7 - \frac{171}{4}x^3 z1^6 + \frac{41}{8}x^4 z1 - 9 x^3 z1^2
+\frac{5}{8}x^2 z2 + \frac{103}{8}x^2 z1^2 - \frac{3}{4}x^3 z2 - \frac{21}{8}x^3 z1 + \frac{15}{2}x^3 z1^5 + \frac{343}{8}x^2 z1^6 - \frac{9}{8}x^2 z1
+\frac{7}{2}x^2 z1^5 - \frac{141}{16}x^2 z1^4 - \frac{85}{8}x^2 z1^3 - 5 z1^5 z2 + 7 z1^4 z2 - \frac{7}{4}z1^3 z2 - \frac{5}{4}z1^2 z2
$$

$$+\frac{1}{4}\,z2\,z1+\frac{19}{2}\,z1^5-\frac{81}{8}\,z1^4+\frac{27}{4}\,z1^3-\frac{13}{8}\,z1^2-\frac{23}{4}\,x^3\,z1^{12}\,z2-\frac{5}{4}\,x^4\,z1^{10}\,z2$$
$$-\frac{33}{2}\,x^3\,z1^{11}\,z2+\frac{3}{4}\,x^4\,z1^9\,z2+\frac{11}{4}\,x^4\,z1^8\,z2+\frac{3}{4}\,x^4\,z1^7\,z2-\frac{25}{4}\,x^4\,z1^6\,z2$$
$$-\frac{5}{4}\,x^4\,z1^5\,z2-\frac{7}{4}\,x^3\,z1^{15}\,z2+\frac{33}{4}\,x\,z1^{17}\,z2+\frac{11}{4}\,x^3\,z1^{14}\,z2+\frac{15}{4}\,x^3\,z1^{13}\,z2$$
$$+\frac{39}{2}\,x^3\,z1^5\,z2+\frac{11}{4}\,x^2\,z1^{15}\,z2-\frac{11}{2}\,x\,z1^{16}\,z2-\frac{41}{8}\,x^2\,z1^{14}\,z2-\frac{219}{8}\,x\,z1^{15}\,z2$$
$$-\frac{7}{2}\,x^2\,z1^{13}\,z2+\frac{93}{4}\,x\,z1^{14}\,z2+\frac{79}{8}\,x^2\,z1^{12}\,z2+\frac{275}{8}\,x\,z1^{13}\,z2+15\,x^3\,z1^{10}\,z2$$
$$+\frac{73}{4}\,x^2\,z1^{11}\,z2-\frac{119}{4}\,x\,z1^{12}\,z2+\frac{43}{2}\,x^3\,z1^9\,z2-\frac{19}{2}\,x^3\,z1^8\,z2-\frac{55}{2}\,x^3\,z1^7\,z2$$
$$+\frac{41}{2}\,x^3\,z1^6\,z2+\frac{11}{4}\,x^3\,z1\,z2-\frac{25}{4}\,x^3\,z1^4\,z2+\frac{39}{4}\,x^3\,z1^2\,z2-\frac{47}{4}\,x^3\,z1^3\,z2$$
$$-\frac{85}{4}\,x^2\,z1^9\,z2+32\,x\,z1^6\,z2-\frac{95}{2}\,x\,z1^{11}\,z2+48\,x\,z1^{10}\,z2+\frac{107}{4}\,x^2\,z1^7\,z2-\frac{9}{4}\,x^2\,z1\,z2$$
$$-\frac{207}{8}\,x\,z1^7\,z2-\frac{49}{2}\,x^2\,z1^{10}\,z2+\frac{33}{2}\,x^2\,z1^8\,z2-\frac{51}{2}\,x^2\,z1^6\,z2-\frac{27}{2}\,x\,z1^4\,z2$$
$$-\frac{13}{8}\,x\,z1^3\,z2+\frac{63}{8}\,x^2\,z1^4\,z2+\frac{53}{4}\,x\,z1^5\,z2-\frac{67}{8}\,x^2\,z1^2\,z2+\frac{19}{4}\,x\,z1^2\,z2$$
$$+\frac{19}{2}\,x^2\,z1^3\,z2+\frac{5}{8}\,x\,z1\,z2+\frac{321}{8}\,x\,z1^9\,z2-\frac{69}{4}\,x^2\,z1^5\,z2-\frac{149}{4}\,x\,z1^8\,z2+\frac{1}{8}\,z1$$
$$+\frac{9}{8}\,x^3+3\,x\,z1^{18}\,z2+\frac{1}{4}\,x^4\,z1^{11}\,z2-\frac{1}{8}\,x\,z1^{21}\,z2+\frac{5}{4}\,x\,z1^{20}\,z2-\frac{33}{8}\,x\,z1^{19}\,z2$$
$$+\frac{1}{4}\,x^3\,z1^{16}\,z2-\frac{3}{8}\,x^2\,z1^{16}\,z2-\frac{1}{16}\,z1^{26}+\frac{7}{8}\,z1^{25}-\frac{19}{4}\,z1^{24}+\frac{47}{4}\,z1^{23}-\frac{157}{16}\,z1^{22}$$
$$-\frac{11}{2}\,z1^{21}-\frac{91}{8}\,z1^{20}+\frac{139}{2}\,z1^{19}-\frac{787}{16}\,z1^{18}-\frac{59}{2}\,z1^{17}-\frac{101}{8}\,x^2\,z1^{17}-\frac{501}{8}\,x\,z1^{18}$$
$$+\frac{9}{8}\,x^4\,z1^{14}-\frac{25}{4}\,x^3\,z1^{15}-\frac{59}{16}\,x^2\,z1^{16}+58\,x\,z1^{17}-\frac{13}{4}\,x^4\,z1^{13}-36\,x^3\,z1^{14}$$
$$+\frac{49}{4}\,x^2\,z1^{15}+\frac{113}{2}\,x\,z1^{16}+2\,x^4\,z1^{12}+\frac{307}{8}\,x^3\,z1^{13}+\frac{305}{8}\,x^2\,z1^{14}+\frac{29}{8}\,x^4\,z1^{11}$$
$$+\frac{271}{8}\,x^3\,z1^{12}-54\,x^2\,z1^{13}-\frac{51}{4}\,x^3\,z1^{11}+\frac{9}{8}\,x^4\,z1^8+\frac{193}{8}\,x^3\,z1^8+\frac{1}{8}\,z1^{22}\,z2$$
$$-\frac{5}{4}\,z1^{21}\,z2+\frac{33}{8}\,z1^{20}\,z2-\frac{13}{4}\,z1^{19}\,z2-\frac{49}{8}\,z1^{18}\,z2+\frac{115}{4}\,z1^{16}\,z2-\frac{57}{4}\,z1^{15}\,z2$$
$$-\frac{53}{2}\,z1^{14}\,z2-\frac{17}{4}\,z1^{13}\,z2+\frac{191}{4}\,z1^{12}\,z2-18\,z1^{11}\,z2-\frac{183}{8}\,z1^{10}\,z2-\frac{35}{4}\,z1^9\,z2$$
$$+\frac{229}{8}\,z1^8\,z2-\frac{39}{4}\,z1^7\,z2-\frac{45}{8}\,z1^6\,z2-\frac{1}{4}\,x\,z2+\frac{1}{16}\,x\,z1^{25}-\frac{7}{8}\,x\,z1^{24}+\frac{19}{4}\,x\,z1^{23}$$
$$-\frac{1}{8}\,x^3\,z1^{20}-\frac{93}{8}\,x\,z1^{22}+\frac{11}{8}\,x^3\,z1^{19}+\frac{3}{16}\,x^2\,z1^{20}+\frac{33}{4}\,x\,z1^{21}-\frac{21}{4}\,x^3\,z1^{18}$$
$$-\frac{17}{8}\,x^2\,z1^{19}+\frac{103}{8}\,x\,z1^{20}+\frac{25}{4}\,x^3\,z1^{17}+\frac{69}{8}\,x^2\,z1^{18}-\frac{7}{2}\,x\,z1^{19}-\frac{1}{8}\,x^4\,z1^{15}$$
$$+\frac{47}{8}\,x^3\,z1^{16}-\frac{831}{8}\,x\,z1^{10}+55\,x\,z1^9-\frac{39}{4}\,x\,z1^7-\frac{335}{8}\,x\,z1^6+\frac{57}{4}\,x\,z1^5+7\,x\,z1^4$$
$$+\frac{1}{4}\,x\,z1^3-\frac{27}{4}\,x\,z1^2+\frac{37}{16}\,x\,z1-\frac{85}{2}\,x\,z1^{15}-\frac{471}{4}\,x\,z1^{14}+\frac{791}{8}\,x\,z1^{13}$$

$$+\frac{153}{2}\,x\,z1^{12}-43\,x\,z1^{11}-\frac{331}{16}\,x^2\,z1^8+\frac{317}{8}\,x\,z1^8-\frac{511}{8}\,z1^{12}+\frac{441}{4}\,z1^{11}$$
$$-\frac{689}{16}\,z1^{10}+\frac{37}{2}\,z1^9+\frac{93}{2}\,z1^7-\frac{291}{16}\,z1^6-36\,z1^{16}+\frac{537}{4}\,z1^{15}-\frac{979}{16}\,z1^{14}-7\,z1^{13}$$
$$-\frac{179}{4}\,z1^8-\frac{1}{8}\,x+x^6-2\,x^5-\frac{3}{8}\,x^4-\frac{11}{16}\,x^2$$

```
> rem(%,f_z,z1);
```

$$x^6-2\,x^5-2\,x^4+2\,x^2-2\,x-1 \tag{11}$$

```
> evala(Factor(f1,{y2,y1}));
```

$$(x+y2+y1-1)\,(x-y2) \tag{12}$$

```
> for i from 1 to 6 do ry||i := -subs(x=0,op(i,facs_y1_y2)); od;
```

$$ry1 := y2$$
$$ry2 := -y1+1-y2$$
$$ry3 := -1+y1^5-2\,y1^4-2\,y1^3+2\,y1-\frac{1}{2}\,y1^5\,y2+\frac{3}{2}\,y1^4\,y2+\frac{1}{2}\,y1^3\,y2-\frac{3}{2}\,y1^2\,y2$$
$$-\frac{5}{2}\,y2\,y1+\frac{1}{2}\,y2$$
$$ry4 := \frac{1}{2}\,y1^5\,y2-\frac{3}{2}\,y1^4\,y2-\frac{1}{2}\,y1^3\,y2+\frac{3}{2}\,y1^2\,y2+\frac{5}{2}\,y2\,y1-\frac{1}{2}\,y2$$
$$ry5 := -y1^5+2\,y1^4+2\,y1^3-2\,y1+2$$
$$ry6 := y1 \tag{13}$$

```
> # check a root by substituting back
> for i from 1 to 6 do evala(subs(x=ry||i, f)); od;
```

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0 \tag{14}$$

```
> # Now we wish to identify an automorphism which maps y1 -> ry4
  # First, find the possible images of y2
> map_y1 := ry4;
```

$$map_y1 := \frac{1}{2}\,y1^5\,y2-\frac{3}{2}\,y1^4\,y2-\frac{1}{2}\,y1^3\,y2+\frac{3}{2}\,y1^2\,y2+\frac{5}{2}\,y2\,y1-\frac{1}{2}\,y2 \tag{15}$$

```
> map_f1 := evala(subs(y1 = map_y1, f1));
```

$$map_f1 := -4+\frac{1}{2}\,x\,y1^5\,y2-\frac{3}{2}\,x\,y1^4\,y2-\frac{1}{2}\,x\,y1^3\,y2+\frac{3}{2}\,x\,y1^2\,y2+\frac{5}{2}\,x\,y2\,y1 \tag{16}$$
$$-\frac{1}{2}\,y1^2\,y2-\frac{1}{2}\,y1^3\,y2+\frac{3}{2}\,y1^4\,y2-\frac{1}{2}\,y1^5\,y2+\frac{1}{2}\,y2\,y1-\frac{1}{2}\,x\,y2+\frac{7}{2}\,y2+y1^5$$
$$-3\,y1^4+2\,y1^2+2\,y1-x+x^2$$

```
> # map_f1 is a factor of f over Q(y1,y2)
  evala(Factor(map_f1, {y2,y1})); # could have plugged in roots
```

$$\left(x-y1^5+2\,y1^4+2\,y1^3-2\,y1+1+\frac{1}{2}\,y1^5\,y2-\frac{3}{2}\,y1^4\,y2-\frac{1}{2}\,y1^3\,y2+\frac{3}{2}\,y1^2\,y2\right. \tag{17}$$
$$\left.+\frac{5}{2}\,y2\,y1-\frac{1}{2}\,y2\right)\,(y1^5-2\,y1^4-2\,y1^3+x+2\,y1-2)$$

```
> # Possible images for y2 are ry1 and ry5
  map_y2 := ry5;
```

$$map_y2 := -y1^5 + 2\,y1^4 + 2\,y1^3 - 2\,y1 + 2 \tag{18}$$

```
> evala(subs(x=map_y2,map_f1));
```

$$0 \tag{19}$$

```
> # Compute the permutation of the roots by mapping them
  for i from 1 to 6 do
      map_ry||i := evala(subs({y1=map_y1,y2=map_y2},ry||i));
      # print(i);
      print(sort(map_ry||i));
      # print("Check if root of f",
              #evala(subs(x=map_ry||i,f)));
      for j from 1 to 6 do
          if eval(map_ry||i - ry||j) = 0 then print(cat(i,"->",j));
  fi;
      od;
  end do:
```

$$-y1^5 + 2\,y1^4 + 2\,y1^3 - 2\,y1 + 2$$

$$1->5$$

$$-1 + y1^5 - 2\,y1^4 - 2\,y1^3 + 2\,y1 - \frac{1}{2}\,y1^5\,y2 + \frac{3}{2}\,y1^4\,y2 + \frac{1}{2}\,y1^3\,y2 - \frac{3}{2}\,y1^2\,y2 - \frac{5}{2}\,y2\,y1 + \frac{1}{2}\,y2$$

$$2->3$$

$$y1$$

$$3->6$$

$$y2$$

$$4->1$$

$$-y1 + 1 - y2$$

$$5->2$$

$$\frac{1}{2}\,y1^5\,y2 - \frac{3}{2}\,y1^4\,y2 - \frac{1}{2}\,y1^3\,y2 + \frac{3}{2}\,y1^2\,y2 + \frac{5}{2}\,y2\,y1 - \frac{1}{2}\,y2$$

$$6->4 \tag{20}$$

```
> # Compute the norm of a random element
> u:=  RandomTools['Generate'](polynom(integer(range=-10..10),{z1,
  z2}));
```

$$u := 6\,z1^5 - 3\,z1^4\,z2 - 3\,z1^3\,z2^2 - 5\,z1^2\,z2^3 + 5\,z2^4\,z1 - 7\,z2^5 - 3\,z1^4 + 5\,z1^3\,z2 - 3\,z1^2\,z2^2 - 2\,z1\,z2^3 - 4\,z2^4 + 4\,z1^3 - z1^2\,z2 + 3\,z1\,z2^2 - 2\,z2^3 - z1^2 + 3\,z2\,z1 + 4\,z2^2 - 10\,z1 - 9\,z2 + 4 \tag{21}$$

```
> norm_u_y2:=evala(expand(subs({z1=y1,z2=y2},u) * subs({z1=y1,z2=
  ry2},u)));
```

$$norm_u_y2 := -6216\,y1^5 + 26491\,y1^4 + 7173\,y1^3 - 24920\,y1^2 + 14911\,y1 + 8918 \tag{22}$$

```
> norm_u_y2_y1:=1: for i from 1 to 6 do
                       norm_u_y2_y1 := evala(expand(norm_u_y2_y1 *
```

```
  subs(y1=ry||i, norm_u_y2)));
                    od:
> norm_u_y2_y1;
```

$$85006691375689613593 \tag{23}$$

```
> ifactor(%);
```

$$(3)^3\,(31483959768773931059) \tag{24}$$

```
> norm_ux_y2:=evala(expand((x - subs({z1=y1,z2=y2},u)) * (x - subs(
  {z1=y1,z2=ry2},u))));
```

$$norm_ux_y2 := 198\,x\,y1^5 - 622\,x\,y1^4 - 6216\,y1^5 + 83\,x\,y1^3 + 26491\,y1^4 + 329\,x\,y1^2 + 7173\,y1^3 + x^2 - 225\,x\,y1 - 24920\,y1^2 - 129\,x + 14911\,y1 + 8918 \tag{25}$$

```
> evala(Factor(norm_ux_y2,{y2,y1}));
```

$$(-59\,y1^5\,y2 + 153\,y1^5 + 167\,y1^4\,y2 - 449\,y1^4 + 9\,y1^3\,y2 - 24\,y1^3 - 122\,y1^2\,y2 + 323\,y1^2 + 77\,y2\,y1 + x - 193\,y1 + 34\,y2 - 111)\,(59\,y1^5\,y2 + 45\,y1^5 - 167\,y1^4\,y2 - 173\,y1^4 - 9\,y1^3\,y2 + 107\,y1^3 + 122\,y1^2\,y2 + 6\,y1^2 - 77\,y2\,y1 + x - 32\,y1 - 34\,y2 - 18) \tag{26}$$

```
> norm_ux_y2_y1:=1: for i from 1 to 6 do
                    norm_ux_y2_y1 := evala(expand(norm_ux_y2_y1
                                          * subs(y1=
  ry||i, norm_ux_y2)));
                    od:
> norm_ux_y2_y1;
```

$$x^{12} + 1328\,x^{11} - 1562800\,x^{10} - 1469503356\,x^9 + 1227670788824\,x^8 - 91305969561384\,x^7 + 2128887428384790\,x^6 + 3533907977081792\,x^5 - 32566324462415152\,x^4 - 870182947064678044\,x^3 + 3369278719489717272\,x^2 + 5606926708871951608\,x + 85006691375689613593 \tag{27}$$

```
> factor(%);
```

$$x^{12} + 1328\,x^{11} - 1562800\,x^{10} - 1469503356\,x^9 + 1227670788824\,x^8 - 91305969561384\,x^7 + 2128887428384790\,x^6 + 3533907977081792\,x^5 - 32566324462415152\,x^4 - 870182947064678044\,x^3 + 3369278719489717272\,x^2 + 5606926708871951608\,x + 85006691375689613593 \tag{28}$$

```
> `?`
```

$$`?` \tag{29}$$

```
>
```