Efficient Algorithms for Computing the Nearest Polynomial with Constrained Roots

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Nearest approximate GCD in the l^2-norm:

Let $f, g \in \mathbb{C}[z]$, both monic, $\deg(f) = m$ and $\deg(g) = n$.

Assuming that $\mathrm{GCD}(f, g) = 1$, find $\tilde{f}, \tilde{g} \in \mathbb{C}[z]$, s.t.

$\mathrm{GCD}(\tilde{f}, \tilde{g})$ is non-trvial and

$\mathcal{N} = \|f - \tilde{f}\|^2 + \|g - \tilde{g}\|^2$ is minimized.

$\|f\|$ denotes a norm of the coefficient vector of f.

The *symbolic* minimum of $\mathcal{N}$ with respect to a common root $\alpha \in \mathbb{C}$ can be obtained in closed-form:

$$\mathcal{N}_{min} = \frac{\overline{f(\alpha)}f(\alpha)}{\sum_{k=0}^{m-1}(\overline{\alpha}\alpha)^k} + \frac{\overline{g(\alpha)}g(\alpha)}{\sum_{k=0}^{n-1}(\overline{\alpha}\alpha)^k}$$

The individual perturbations of the coefficients of f and g are

$$f_i - \tilde{f}_i = \frac{(\overline{\alpha})^i f(\alpha)}{\sum_{k=0}^{m-1}(\overline{\alpha}\alpha)^k} \quad \text{and} \quad g_j - \tilde{g}_j = \frac{(\overline{\alpha})^j g(\alpha)}{\sum_{k=0}^{n-1}(\overline{\alpha}\alpha)^k}$$

for $0 \le i \le m-1$ and $0 \le j \le n-1$, respectively.

Reduced Problem: Given $f \in \mathbb{C}[z]$ and $\alpha \in \mathbb{C}$. Find $\tilde{f} \in \mathbb{C}[z]$, s.t.

$$\tilde{f}(\alpha) = 0, \quad \text{and} \quad \|f - \tilde{f}\| = \min .$$

Let

$$\begin{aligned}
f(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\
\tilde{f}(z) &= (z - \alpha) \sum_{k=0}^{n-1} u_k z^k \\
&= u_{n-1} z^n + (u_{n-2} - \alpha) z^{n-1} + (u_{n-3} - \alpha u_{n-2}) z^{n-2} + \\
&\qquad \cdots \quad + (u_0 - \alpha u_1) z - \alpha u_0
\end{aligned}$$

In terms of linear algebra:

$$\|f - \tilde{f}\| = \min_{\mathbf{u} \in \mathbb{C}^n} \|\mathbf{P}\mathbf{u} - \mathbf{b}\| \tag{1}$$

$$\begin{aligned} \mathbf{b} &= [a_0, \ldots, a_{n-1}, a_n]^{tr} \in \mathbb{C}^{n+1} \\ \mathbf{u} &= [u_0, \ldots, u_{n-1}]^{tr} \in \mathbb{C}^n \end{aligned}$$

$$\mathbf{P} = \begin{bmatrix} -\alpha & & & \\ 1 & -\alpha & & \\ & \ddots & \ddots & \\ & & 1 & -\alpha \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{(n+1)\times n} \tag{2}$$

(1) is an over-determined linear system of equations.

LP problem, if $\|\cdot\|$ is the $\begin{cases} l^\infty & \text{norm, or} \\ l^1 & \text{norm} \end{cases}$

LS problem, if $\|\cdot\|$ is the l^2 norm.

Solutions for the l^2-norm in closed form:

$$\mathcal{N}_{min}(\alpha) = \|f - \tilde{f}\|^2 = \frac{\overline{f(\alpha)}\, f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k} \qquad f_j - \tilde{f}_j = \frac{(\overline{\alpha})^j f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k}, \; 0 \le j \le n$$

Constraining a Root Locus to a Curve

Let Γ be a piecewise smooth curve with finitely many segments, each having a parametrization $\gamma_k(t)$ in a single real parameter t.

For a given polynomial $f \in \mathbb{C}[z]$, we want to find a minimally perturbed polynomial $\tilde{f} \in \mathbb{C}[z]$ that has (at least) one root on Γ.

Parametric Minimization

We substitute the parametrization $\gamma_k(t)$ for the indeterminate α in $\mathcal{N}_{min}(\alpha)$. The resulting expression is a function in $t \in \mathbb{R}$.

It attains its minima at its *stationary* points. We have to compute the *real* roots of the derivative.

The derivative of the norm-expression is determined *symbolically*, the roots can be computed numerically.

Algorithm C

Input: $f \in \mathbb{C}[z]$, and a curve Γ.

Output: $\tilde{f} \in \mathbb{C}[z]$, and $\tau \in \mathbb{R}$, s.t. $\tilde{f}(\gamma_k(\tau)) = 0$ for some segment of Γ, and $\|f - \tilde{f}\|_2 = \min$.

($\mathbf{C}_1$) For each segement of Γ:

- ($\mathbf{C}_{1.1}$) Substitute $\gamma_k(t)$ for α in the symbolic minimum $\mathcal{N}_{min}(\alpha) \mapsto N(t)$.
- ($\mathbf{C}_{1.2}$) Symbolically determine the derivative $N'(t)$.
- ($\mathbf{C}_{1.3}$) Compute the *real* roots (of the numerator) of $N'(t)$. Select the one that minimizes $N(t)$.

($\mathbf{C}_2$) From all $N(\tau_k)$ of step ($\mathbf{C}_{1.3}$) determine the minimum $N(\tau)$.

($\mathbf{C}_3$) Compute the perturbations δ_j. Return $\tilde{f}$, k, and τ.

Computing the Radius of Stability in the l^2-Norm

Definition: Let $\mathcal{D} \subset \mathbb{C}$ be an open, and convex domain of the complex plane. The polynomial $f \in \mathbb{C}[z]$ is called $\mathcal{D}$-stable, if all its roots are located within $\mathcal{D}$.

Special cases:
- the left half-plane: *Hurwitz* stability
- the open unit-disc: *Schur* stability

Given a $\mathcal{D}$-stable polynomial f, how much can we perturb its coefficients such that the perturbed polynomial is still $\mathcal{D}$-stable?

If we have a (piecewise) real parametrization of the boundary $\partial\mathcal{D}$ then we can apply our algorithm to find a *nearest unstable* polynomial.

Theorem: Let $f \in \mathbb{C}[z]$ be $\mathcal{D}$-stable, and

let $\hat{f} \in \mathbb{C}[z]$ be an unstable polynomial,
such that $\|f - \hat{f}\| = \varepsilon$, where $\varepsilon \in \mathbb{R}, \varepsilon > 0$.

Then, there exists $\tilde{f} \in \mathbb{C}[z]$ and $\zeta \in \partial\mathcal{D}$ such that

$\|f - \tilde{f}\| \leq \varepsilon$ and $\tilde{f}(\zeta) = 0$.

Example (of a monic polynomial)

$$\begin{aligned} f(z) &= z^3 + (2.41 - 3.50\,\mathbf{i})z^2 + (2.76 - 5.84\,\mathbf{i})z \\ &\quad -1.02 - 9.25\,\mathbf{i} \end{aligned}$$

is Hurwitz.

Root locations: $-1.04 + 3.10\,\mathbf{i}$, $-.99 - 1.30\,\mathbf{i}$, $-.37 + 1.70\,\mathbf{i}$

Nearest unstable polynomial:

$$\begin{aligned} \tilde{f}(z) &= z^3 + (2.7037 - 3.1492\,\mathbf{i})z^2 + (2.5740 - 5.6842\,\mathbf{i})z \\ &\quad -1.1026 - 9.3486\,\mathbf{i}\,. \end{aligned}$$

Radius of stability in the l^2-norm: 0.533567.

Conclusions

The constraining of roots to prescribed locations of the complex plane leads to linear minimization problems.

If minimality is expressed in the Euclidean norm, then the associated least square problem can be solved *symbolically*.

For real coefficients and a single real root, the min-max of the perturbation can also be expressed in closed form.

$$\mathcal{N}_\infty(\alpha) \stackrel{def}{=} \|f - \tilde{f}\|_\infty = \left| \frac{f(\alpha)}{\sum_{k=0}^{n} |\alpha^k|} \right|$$

Hybrid algorithms, for constraining a single root to a curve, permit computation of the radius of stability in the Euclidean norm.

Future Activities

Investigating aspects of the practical implementation of the algorithms

Deriving symbolic expressions for a wider variety of algebraic curves in the case of real coefficients

Applying parametric minimization to other problems

Applying our methods to sensitivity analysis of ill-conditioned problems involving polynomial roots