Efficient Algorithms for Computing the Nearest Polynomial with Constrained Roots*

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Abstract

Continuous changes of the coefficients of a polynomial move the roots continuously. We consider the problem finding the minimal perturbations to the coefficients to move a root to a given locus, such as a single point, the real or imaginary axis, the unit circle, or the right half plane. We measure minimality in both the Euclidean distance to the coefficient vector and maximal coefficient-wise change in absolute value (infinity norm), either with entirely real or with complex coefficients. If the locus is a piecewise parametric curve, we can give efficient, i.e., polynomial time algorithms for the Euclidean norm; for the infinity norm we present an efficient algorithm when a root of the minimally perturbed polynomial is constrained to a single point. In terms of robust control, we are able to compute the radius of stability in the Euclidean norm for a wide range of convex open domains of the complex plane.

1 Introduction

Recent results in robust stability can be categorized by two different ways of looking at the same problem: on the one hand, following the landmark paper of Karithonov [13, 15, 14], several *tests* for stability of a *given* family of polynomials have been devised (see [1, 24]). On the other hand, research focuses on the determination of the *radius of stability* for a given nominal polynomial and a certain norm in coefficient space. In this area, notably the work of Tsypkin and Polyak [25], Desages et al. [5], and Kogan [16] provides necessary and sufficient conditions for the range of robust stability in a (weighted) l^p norm. However, these criteria allow actual *computation* of the stability radius only in a few special cases, as they leave infinite one- or two-dimensional sets to be searched for the minimum. Frequency domain plots seem to be the method of choice for more general applications (see [10]). In this paper, we present a method for computing the radius of stability in the l^2 norm for univariate polynomials with either complex or real coefficients. In terms of flavor, it is closest to the ideas presented in [19], and especially in [20]. Although the latter paper addresses the more general problem of unspecified affine coefficient perturbations in several norms, it is limited to the case of real coefficients. As is shown in [3] and [2], their method is equivalent to computing structured singular values for a special class of rank-one problems. The perturbations in our approach are more closely tied to root locations. As a consequence, searching along the contour ([20]), or along a radial ([19]) is reduced to computing the real solutions of an algebraic equation. The method is applicable for a wide range of stability domains $\mathcal{D} \subset \mathbb{C}$. Among them are the familiar cases of Hurwitz and Schur stability.

Our method is based on a recent result for the *nearest approximate greatest common divisor* by Karmarkar and Lakshman [11, 12]. It makes use of the technique of *parametric minimization*, and belongs to the relatively new class of *hybrid* symbolic-numeric algorithms. These algorithms combine the deductive power of modern computer algebra systems with the speed and reliability of numerical packages.

After some notation and theoretical background, we show how to compute the coefficients of the (or a) nearest unstable polynomial for a given stable, monic polynomial with complex coefficients. We later discuss the case of real coefficients, as well as how to deal with degree-drops when allowing the highest-order coefficient to be perturbed. Particular attention is paid to the special case of Hurwitz and Schur stability.

2 Preliminaries

In the following, $\mathbb{C}[z]$ and $\mathbb{R}[z]$ are the rings of univariate polynomials over the complex numbers and real numbers, respectively. Bold letters should indicate vectors and matrices; a bold $\mathbf{i}$ stands for the imaginary unit.

If not stated otherwise, $\|.\|$ denotes the l^2 vector norm. For the sake of notational simplicity, we keep it unweighted at that point; however, all statements are still valid for a weighted l^2-norm. The norm expression $\|f\|$ for a polynomial f is the norm of its coefficient vector.

The operator * applied to matrices and vectors is the Hermite transposition, while ^{tr} is used for the regular transposition.

Let $\mathcal{D} \subset \mathbb{C}$ be an open, convex domain of the complex plane. The polynomial $f \in \mathbb{C}[z]$ (or $f \in \mathbb{R}[z]$) is called $\mathcal{D}$-*stable* if all its roots are located within $\mathcal{D}$. Our goal is to

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derive a method for finding the nearest unstable polynomial $\tilde{f}$ for a given $\mathcal{D}$-stable polynomial f. For simplicity again, we restrict ourselves to fixed-degree polynomials, and without loss of generality to monic polynomials in particular. We will discuss the more general case at the end of this article. The basis of our method is the following theorem, which could also be called the *inverse zero-exclusion principle*:

Theorem 1 *Let $f \in \mathbb{C}[z]$, monic be $\mathcal{D}$-stable ($\mathcal{D}$ as defined above), and let $\hat{f} \in \mathbb{C}$, monic be an unstable polynomial of the same degree as f such that $\|f - \hat{f}\| = \epsilon$, where $\epsilon \in \mathbb{R}, \epsilon > 0$. Then, there exists $\tilde{f} \in \mathbb{C}[z]$ and $\zeta \in \partial\mathcal{D}$ such that $\|f - \tilde{f}\| \leq \epsilon$ and $\tilde{f}(\zeta) = 0$.*

PROOF: For Hurwitz stability, see also Lemma 3.2 in [16]. For $t \in \mathbb{R}$ and $0 < t \leq 1$, we define the polynomials f_t by

$$f_t(z) = f(z) + t \cdot (\hat{f}(z) - f(z)) .$$

Because of our assumptions, all f_t have the same degree as f. For a generic vector norm, the expression $\|f - f_t\| = t \cdot \|\hat{f}(z) - f(z)\|$ is a strictly increasing function in t. Because $\hat{f}$ is unstable, it has one or more roots outside or on the boundary of $\mathcal{D}$. By virtue of the continuous dependence of the roots on the coefficients (see [18]), there must be $\tau \in (0, 1]$ and $\zeta \in \partial\mathcal{D}$ such that $f_\tau(\zeta) = 0$ and $\|f - f_\tau\| \leq \epsilon$. □

In other words: the roots of fixed-degree polynomials cannot "jump" out of the domain $\mathcal{D}$ without crossing the boundary $\partial\mathcal{D}$. If we allow perturbations of the leading coefficient, we have to be more careful in case the domain is unbounded (see section 6). Therefore, for any unstable polynomial we can find a polynomial with a root on $\partial\mathcal{D}$ that is as close (in coefficient space) as the given one. I.e., it suffices to look for such polynomials to find a nearest unstable one.

3 The Nearest Polynomial with a Constrained Root

First, we want to solve the following reduced problem:

Problem 1 *Given $f \in \mathbb{C}[z]$ monic, $\deg(f) = n$, For a complex indeterminate α, find $\tilde{f} \in \mathbb{C}[z]$, monic, such that*

$$\tilde{f}(\alpha) = 0 , \quad \text{and} \quad \|f - \tilde{f}\| = \min .$$

Let

$$f \in \mathbb{C}[z] , \quad f(z) = \sum_{k=0}^{n} a_k z^k , \quad \text{monic.}$$

For the complex indeterminate α, we define the perturbed, monic polynomial $\tilde{f} \in \mathbb{C}[z]$ such that $\tilde{f}(\alpha) = 0$:

$$\begin{aligned} \tilde{f}(z) &= (z - \alpha) \sum_{k=0}^{n-1} u_k z^k \\ &= z^n + (u_{n-2} - \alpha) z^{n-1} + (u_{n-3} - \alpha u_{n-2}) z^{n-2} + \\ &\quad \cdots + (u_0 - \alpha u_1) z - \alpha u_0 \\ &= z^n + \sum_{k=1}^{n-1} (u_{k-1} - \alpha u_k) z^k - \alpha u_0 , \end{aligned}$$

where $u_k \in \mathbb{C}$ and $u_{n-1} = 1$. Furthermore, we define the perturbation $\delta = f - \tilde{f}$, a polynomial of degree $n - 1$. Now, let

$$\begin{aligned} \mathbf{b} &= [a_0, \ldots, a_{n-2}, a_{n-1} + \alpha]^{tr} \in \mathbb{C}^n , \quad \text{and} \\ \mathbf{u} &= [u_0, \ldots, u_{n-2}]^{tr} \in \mathbb{C}^{n-1} . \end{aligned}$$

In order to minimize $\|\delta\|$, we have to solve the following *parametrized* least squares problem:

$$\|\delta\| = \min_{\mathbf{u} \in \mathbb{C}^{n-1}} \|\mathbf{P}\mathbf{u} - \mathbf{b}\| , \tag{1}$$

where the rectangular matrix

$$\mathbf{P} = \begin{bmatrix} -\alpha & & & 0 \\ 1 & -\alpha & & \\ & \ddots & \ddots & \\ 0 & & 1 & -\alpha \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{n \times (n-1)} \tag{2}$$

represents the polynomial multiplication operator for $(z-\alpha)$. Due to the monicity condition, the leading coefficients are not part of the equation. However, we have to add $u_{n-1}\alpha = \alpha$ to the last element of the constant vector $\mathbf{b}$ to account for this omission. We can state the following lemma:

Lemma 1 *For any choice $\alpha \in \mathbb{C}$, the least squares problem (1) has a unique solution $\mathbf{u}_m(\alpha) \in \mathbb{C}^{n-1}$.*

PROOF: For any $\alpha \in \mathbb{C}$ the column vectors of $\mathbf{P}$ are linearly independent, thus $\mathbf{P}$ has full rank. □

We can derive an explicit formula for the parametric minimum. It will enable us to compute the perturbation of each coefficient as well as the root α if we are given suitable constraints (such as domain boundaries) for α. The following theorem summarizes the results for complex coefficients:

Theorem 2 *At the minimum $\mathbf{u}_m(\alpha)$ the square of the norm of the minimal perturbation is*

$$\mathcal{N}_m(\alpha) = \|\delta\|^2 = \frac{\overline{f(\alpha)} f(\alpha)}{\sum_{k=0}^{n-1} (\bar{\alpha}\alpha)^k} , \tag{3}$$

and the perturbation of the j-th coefficient is

$$\delta_j = \frac{(\bar{\alpha})^j f(\alpha)}{\sum_{k=0}^{n-1} (\bar{\alpha}\alpha)^k} , \tag{4}$$

for $0 \leq j \leq n - 1$, and assuming that $0^0 = 1$.

PROOF: The formulas can be obtained by either explicitly solving the normal equations, or equivalently, minimizing a quadratic form (see [11]), or via a Lagrange multiplier approach (see [12]). A proof based on vector geometry is given in appendix 8.1. □

The minimum $\mathcal{N}_m(\alpha)$ as a function of α is real-valued and non-negative. Its denominator is the determinant of the matrix $\mathbf{Q} = (\mathbf{P}^*)\mathbf{P}$, and is strictly positive.

4 The Nearest Unstable Polynomial – Complex Coefficients

By combining theorem 1 and 2, we are able to develop an algorithm for computing a nearest unstable polynomial in the l^2-norm. To accomplish that, we need a suitable parametrization $\gamma(t)$ of the domain boundary $\partial\mathcal{D}$, where $t \in I \subset \mathbb{R}$. We then substitute this parametrization for the indeterminate α in the formula for $\mathcal{N}_m(\alpha)$, resulting in an expression $\mathcal{N}_m(\gamma(t))$ as a function of the real parameter t. The minima can be found at the stationary points of this function, i.e., we have to determine the *real* solutions of the equation:

$$\frac{d\mathcal{N}_m}{dt} = \frac{d\mathcal{N}_m}{d\gamma}\frac{d\gamma}{dt} = 0\,. \tag{5}$$

For each solution τ_k, we obtain a root $\zeta_k = \gamma(\tau_k)$ of an unstable polynomial with $\zeta_k \in \partial\mathcal{D}$. We have to select the one with the smallest $\mathcal{N}_m(\gamma(\tau_k))$.

We illustrate the procedure in the following example:

Example 1 *The polynomial*

$$\begin{aligned} f(z) &= z^3 + (2.41 - 3.50\,\mathbf{i})z^2 + (2.76 - 5.84\,\mathbf{i})z \\ &\quad -1.02 - 9.25\,\mathbf{i} \end{aligned}$$

is Hurwitz, with approximate roots: $-1.04 + 3.10\,\mathbf{i}$, $-.99 - 1.30\,\mathbf{i}$, *and* $-.37 + 1.70\,\mathbf{i}$. *The boundary of the left half-plane can be parametrized by* $\gamma(t) = \mathbf{i}t$, *where* $t \in \mathbb{R}$. *Consequently,*

$$\begin{aligned} \mathcal{N}_m(\gamma(t)) &= (t^6 - 7.00t^5 + 12.5381t^4 + 9.6712t^3 \\ &\quad -18.1104t^2 - 62.9736t + 86.6029) \\ &\quad /(t^4 + t^2 + 1)\,. \end{aligned}$$

The numerator of the derivative is a monic polynomial of degree nine. It has three real roots (rounded to 6 digits): $\tau_1 = -1.84729$, $\tau_2 = -.248977$, *and* $\tau_3 = 1.88617$. $\mathcal{N}_m(\gamma(\tau_k))$ *evaluates to* 25.9376, 94.8227, *and* .284693 *respectively. Therefore,* τ_3 *leads to minimal perturbations. The* l^2*-norm distance (radius of stability) is the square root of* $\mathcal{N}_m$, *namely* .533567. *Using (4), we can compute the coefficients of the perturbed polynomial* $\tilde{f}$:

$$\begin{aligned} \tilde{f}(z) &= z^3 + (2.7037 - 3.1492\,\mathbf{i})z^2 + (2.5740 - 5.6842\,\mathbf{i})z \\ &\quad -1.1026 - 9.3486\,\mathbf{i}\,. \end{aligned}$$

The roots moved to $-1.6472 + 2.5328\,\mathbf{i}$, $-1.0566 - 1.2698\,\mathbf{i}$, *and* $1.8862\,\mathbf{i}$ *respectively (see figure 1).*

We are now ready to give a formal description of the algorithm and its requirements. The problem, we want to solve is the following:

Problem 2 *Let* $\mathcal{D} \subset \mathbb{C}$ *be a domain as described in section 2, whose boundary* $\partial\mathcal{D}$ *has a piece-wise smooth, closed parametrization (Jordan-curve), where each segment is a map* γ *from an interval* $I \subset \mathbb{R}$ *onto* $\mathbb{C} \cup \{\infty\}$. *Furthermore, let* f *be a monic* $\mathcal{D}$*-stable polynomial with complex coefficients. Then find a monic polynomial* $\tilde{f}$ *such that* $\|f - \tilde{f}\| = \min$ *and* $\tilde{f}$ *is* $\mathcal{D}$*-unstable, where the computations are performed within some given (numerical) precision.*

Based on the discussion at the beginning of the section, we propose the following algorithm to compute a nearest unstable polynomial:

Algorithm C:

C1: For each segment of $\partial\mathcal{D}$ with parametrization $\gamma : I \to \mathbb{C} \cup \{\infty\}$, substitute $\gamma(t)$ for α in the symbolic minimum $\mathcal{N}_m(\alpha)$.

C1.1: Determine the derivative of $\mathcal{N}_m(\gamma(t))$ symbolically.

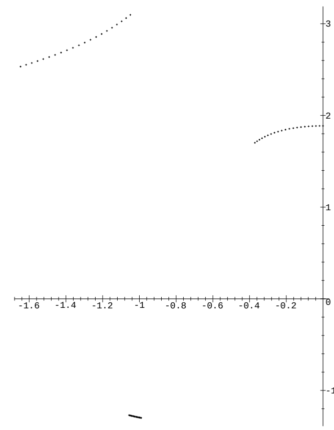

Figure 1:

C1.2: Determine all *real* solutions τ_k to equation (5); evaluate $\mathcal{N}_m(\gamma(\tau_k))$.

C2: From the values for $\mathcal{N}_m$ computed in all steps C1.2, select the minimum. Compute the perturbed coefficients, and return $\tilde{f}$.

Remarks:

- Certain parametrizations have singular points which have to be treated separately, as additional segments. E.g., the popular parametrization of the unit-circle:

$$\gamma(t) = \frac{t - \mathbf{i}}{t + \mathbf{i}}\,,$$

  for $-\infty < t < \infty$ leaves an infinitesimal gap at 1. In this case, we have to evaluate $\mathcal{N}_m(1)$, and compare the result with the values obtained in C1.2. Another solution would be to use two half-circles whose parameter t only runs from -1 to $+1$.

- Computer algebra systems are capable of computing the derivative for a wide range of parametrizations *symbolically*. However, numerical techniques (difference methods) could be used for cases there no closed form for γ is available.

- Solving the equations in C1.2 is computationally the most expensive step of the algorithm. Numerical root-finders are most likely to be superior compared to implementations based on exact arithmetic.

- Complexity-wise, all steps are in polynomial time.

5 The Nearest Unstable Polynomial – Real Coefficients

If $f \in \mathbb{R}[z]$, monic, and we allow only real perturbations of the coefficients, we have to distinguish between real roots and pairs of complex conjugates. Real roots lie in the intersection of the domain boundary $\partial\mathcal{D}$ and the real axis. We can use algorithm C for real values to find roots on that segment of $\partial\mathcal{D}$. In the second case, we have to find a perturbed polynomial with a pair of complex conjugate roots whereof (at least) one lies on the boundary $\partial\mathcal{D}$. From this it is clear that the stability domain $\mathcal{D}$ has to have a non-empty intersection with the real axis if the coefficients of the perturbed polynomial have to remain real (we could relax this condition by allowing domains with two separate components, where one component has a non-empty intersection with the mirror-image with respect to the real axis of the other component). The following small example shows the different cases for Schur-stability:

Example 2 *The polynomial $f(z) = z^2 - 0.1z - 0.3$ has (real) roots -0.6 and 0.5; both are inside the unit-circle. By perturbing the coefficients of f, one root can become either -1 or $+1$, or both can be a pair of complex conjugates located on the unit-circle. Using (3) from section 2, we have $\mathcal{N}_m(-1) = 0.32$ and $\mathcal{N}_m(1) = 0.18$. Alternatively, the perturbed polynomial $\tilde{f}$ would have roots α and $\bar{\alpha}$, where $\alpha\bar{\alpha} = 1$, i.e., $\tilde{f}(z) = (z-\alpha)(z-\bar{\alpha}) = z^2 - 2az + 1$, and $a = \Re(\alpha) \in \mathbb{R}$. Perturbing the constant coefficient (-0.3) to a value of $+1$ would already give us a contribution of $1.3^2 = 1.69$ towards $\mathcal{N}$. Therefore, moving one root to $+1$ will result in the minimal perturbation. $\tilde{f} = z^2 - 0.4z - 0.6$ has another real root at -0.6.*

In general, we have to perturb f such that $\tilde{f}$ has a factor $(z-\alpha)(z-\bar{\alpha}) = z^2 - 2az + a^2 + b^2$, where $a = \Re(\alpha) \in \mathbb{R}$ and $b = \Im(\alpha) \in \mathbb{R}$. If we set $s = -2a$ and $q = a^2 + b^2$, we are lead to the least square problem, analogous to (1):

$$\|\delta\| = \min_{\mathbf{v}\in\mathbb{R}^{n-2}} \|\mathbf{R}\mathbf{v} - \mathbf{c}\|\,, \qquad (6)$$

where

$$\begin{aligned}
\mathbf{v} &= [v_0, \ldots, v_{n-3}]^{tr} \in \mathbb{R}^{n-2}\,,\\
\tilde{f}(z) &= z^n + (v_{n-3} + s)z^{n-1} + (v_{n-4} + sv_{n-3} + q)z^{n-2}\\
&\quad +(v_{n-5} + sv_{n-4} + qv_{n-3})z^{n-3} + \cdots\\
&\quad +(v_0 + sv_1 + qv_2)z^2 + (sv_0 + qv_1)z + qv_0\,,\\
\mathbf{c} &= [a_0, \ldots, a_{n-2} - q, a_{n-1} - s] \in \mathbb{R}^n\,, \text{ and}\\
n &= \text{degree}(f) \geq 3\,.
\end{aligned}$$

The tridiagonal $n \times (n-2)$-matrix

$$\mathbf{R} = \begin{bmatrix} q & & & & 0 \\ s & q & & & \\ 1 & s & q & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & s & q \\ 0 & & & 1 & s \\ & & & & 1 \end{bmatrix} \qquad (7)$$

has again full rank, thus warranting a unique solution for (6). Most computer algebra systems allow users to specify additional properties of variables. For symbolic derivations, the fact that q is positive can be of advantage.

Unfortunately, the symbolic expressions are not as simple as in the complex case. However, the symbolic minimum can still be computed for a given problem, such that the algorithm from the previous section remains essentially the same. For completeness, we present the entire algorithm for polynomials with real coefficients:

Algorithm R:

R1: Derive the symbolic solution $\mathbf{v}_m(s,q)$ to (6) for the given polynomial $f \in \mathbb{R}[z]$. Derive $\mathcal{N}_m(s,q) = (\mathbf{R}\mathbf{v}_m - \mathbf{c})^{tr}(\mathbf{R}\mathbf{v}_m - \mathbf{c})$.

R2: For each segment of $\partial\mathcal{D}$ with parametrization $\gamma\colon I \to \mathbb{C} \cup \{\infty\}$, substitute $-2\Re(\gamma(t))$ for s, and $|\gamma(t)|^2 = \Re(\gamma(t))^2 + \Im(\gamma(t))^2$ for q in the symbolic minimum $\mathcal{N}_m(s,q)$, resulting in an expression $\mathcal{N}_m(t)$.

R2.1: Determine the derivative $\mathcal{N}_m'(t)$ of $\mathcal{N}_m(t)$ symbolically.

R2.2: Determine all *real* solutions τ_k to $\mathcal{N}_m'(t) = 0$; evaluate $\mathcal{N}_m(\tau_k)$.

R3: Use algorithm C from the previous section to compute $\mathcal{N}_m$ for the intersection points of $\partial\mathcal{D}$ with the real axis.

R4: From the values for $\mathcal{N}_m$ computed in all steps R2.2, and step R3 select the minimum. Determine the perturbations using expressions from step R1, or formulas (4) respectively. Return $\tilde{f}$.

For *Hurwitz* stability, we can again give explicit formulas. The special form of the domain boundary (imaginary axis) reduces (6) to a problem in a single parameter. In this case, $\tilde{f}$ will have a factor $(z - \mathbf{i}t)(z + \mathbf{i}t) = z^2 + t^2$, where $t \in \mathbb{R}$, i.e., $s = 0$ and $q = t^2$. We summarize the results in the following theorem:

Theorem 3 *Let $f \in \mathbb{R}[z]$ be Hurwitz. We define the polynomials $g, h \in \mathbb{R}[x]$, such that $f(z) = g(z^2) + z \cdot h(z^2)$, and substitute x for z^2. The nearest polynomial having at least one root on the imaginary axis at $z = \mathbf{i}t$ is given by the following perturbations of the coefficients of f:*

a) *If $n = \text{degree}(f)$ is even, and $m = n/2$, then*

$$\begin{aligned}
\delta_j &= \frac{\left(-t^2\right)^{j/2} g\left(-t^2\right)}{\sum_{k=0}^{m-1} t^{4k}}\,, \qquad j \text{ even},\\
\delta_j &= \frac{\left(-t^2\right)^{(j-1)/2} h\left(-t^2\right)}{\sum_{k=0}^{m-1} t^{4k}}\,, j \text{ odd},\\
\mathcal{N}_m(t) &= \frac{g^2\left(-t^2\right) + h^2\left(-t^2\right)}{\sum_{k=0}^{m-1} t^{4k}}\,.
\end{aligned} \qquad (8)$$

b) *If $n = \text{degree}(f)$ is odd, and $m = (n-1)/2$, then*

$$\begin{aligned}
\delta_j &= \frac{\left(-t^2\right)^{j/2} g\left(-t^2\right)}{\sum_{k=0}^{m} t^{4k}}\,, \qquad j \text{ even},\\
\delta_j &= \frac{\left(-t^2\right)^{(j-1)/2} h\left(-t^2\right)}{\sum_{k=0}^{m-1} t^{4k}}\,, \quad j \text{ odd},\\
\mathcal{N}_m(t) &= \frac{g^2\left(-t^2\right)}{\sum_{k=0}^{m} t^{4k}} + \frac{h^2\left(-t^2\right)}{\sum_{k=0}^{m-1} t^{4k}}\,.
\end{aligned} \qquad (9)$$

Proof: A proof of this theorem can be found in appendix 8.2. □

Remarks:

- In order to compute the actual minimum, one still has to execute steps R2.2, R3, and R4 of algorithm R.

We could take the formal derivation even further, and explicitly write down the (polynomial) equations in t, that one has to solve. However, this can easily be done by a computer algebra system, and does not provide further insights into the problem.

- Formulas (8) and (9) are consistent with the Hermite-Biehler theorem ([7, 17]). They represent the fact that g and h are perturbed independently, such that they have $-t^2$ as a common root. The norm expressions $\mathcal{N}_m$ are the ones we could derive from the formulas for the approximate GCD problem of two polynomials in the l^2-norm (see [11, 12]) by substitution.

In the case where $\mathcal{D}$ is the open unit-disc (*Schur* stability), the matrix $\mathbf{R}$ of (7) is also depending on a single parameter, namely s, as $q = 1$. Here however, the analogue of the *interlacing property* (as in the Hermite-Biehler theorem) does not apply to polynomials that are constructed from the coefficients of f directly (see [17]). Therefore, the derivation of explicit formulas for $\mathcal{N}_m(s)$ would not save us significant compute time over a direct solution of the least squares problem.

6 Extensions and Generalizations

6.1 Other Norms

The minimization problem (1) is stated in a form that is norm-independent. In fact the reduced problem, where α is a given constant, can also be solved in the l^∞ and l^1-norm. Algorithms for finding approximate solutions to inconsistent systems of linear equations in these norms are well known (see e.g., [23], [8, chapter on least squares] or [4], and the literature therein), and they all can be made to run in polynomial time. The basic algorithms are also given in Appendix 8.3. However, the involved linear programming substeps seem more difficult to use in *parametric* minimization. The methods of Appendix 8.3 are applicable in a straightforward manner to *real* roots and *real* coefficients. In the complex case we separate real and imaginary parts by applying the following decomposition: if $\alpha = a + \mathbf{i}b,\ a, b \in \mathbb{R}$ and

$$\mathbf{u} = [v_0 + \mathbf{i}w_0, \ldots, v_{n-2} + \mathbf{i}w_{n-2}]^{tr},$$

then in (2) $\mathbf{P} = \mathbf{A} + \mathbf{i}\mathbf{B}$, where

$$\mathbf{A} = \begin{bmatrix} -a & & & 0 \\ 1 & -a & & \\ & \ddots & \ddots & \\ 0 & & 1 & -a \\ & & & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -b & & & 0 \\ 0 & -b & & \\ & \ddots & \ddots & \\ 0 & & 0 & -b \\ & & & 0 \end{bmatrix}$$

are real $n \times (n-1)$ matrices. If we define

$$\mathbf{v} = [v_0, \ldots, v_{n-2}]^{tr} \text{ and } \quad \mathbf{w} = [w_0, \ldots, w_{n-2}]^{tr},$$

then we can replace the product $\mathbf{Pu}$ by

$$\begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix},$$

and set up the minimization problem accordingly.

6.2 Weighted l^2-norms

If we assign positive weights w_k to each component in computing the norm, i.e.,

$$\|f\|_w = \left(\sum_{k=0}^{n-1} w_k \bar{a}_k a_k\right)^{1/2}, \quad \text{for} \quad f(z) = z^n + \sum_{k=0}^{n-1} a_k z^k,$$

then $\mathcal{N}_m(\alpha)$ in (3) becomes (see [12]):

$$\mathcal{N}_m(\alpha) = \frac{\overline{f(\alpha)} f(\alpha)}{\sum_{k=0}^{n-1} w_k^{-1} (\bar{\alpha}\alpha)^k}.$$

6.3 Perturbing the Leading Coefficient

Basically, there is no problem by letting the leading coefficient be subject to perturbation, i.e., to admit non-monic polynomials. The only difficulty arises when the leading coefficient *vanishes* in this process, and the degree of the perturbed polynomial drops. As the leading coefficient approaches zero, (at least) one of the roots goes to ∞. Theorem 1 still holds if we include the notion of points at ∞ (in the projective plane sense). This is only necessary for unbounded domains, like the open left half-plane in the Hurwitz case, where ∞ can be on the boundary $\partial\mathcal{D}$. As long as the domain $\mathcal{D}$ is bounded our algorithms can be extended to the general case in a straightforward way. Roots moving out to ∞ have to cross the domain boundary first in that case. The following example shows that for unbounded domains the minimal perturbation can be the one causing a drop in degree. If we restrict ourselves to an affine view of the problem, and if we add the constraint that the leading coefficient must not vanish, there might be no minimal solution at all.

Example 3 *The linear polynomial $f(z) = z + 2$ is Hurwitz. If we look at real perturbations, the nearest unstable polynomial of degree one is $\tilde{f}_1(z) = z$, with $\|f - \tilde{f}_1\| = 2$. By dropping the leading term, we end up with the degree-zero polynomial $\tilde{f}_0(z) = 2$. The norm of the perturbation $\|f - \tilde{f}_0\| = 1$ is obviously smaller than the one we would get from the least squares solution. Adding the additional constraint that the leading coefficient of $\tilde{f}$ is not allowed to vanish leads to the family of unstable polynomials $\tilde{f}_\epsilon(z) = -\epsilon z + 2$, where $\epsilon \in \mathbb{R}$, $\epsilon > 0$. The norm of the perturbation $\|f - \tilde{f}_\epsilon\| = 1 + \epsilon$ does not have a minimum with respect to $\epsilon > 0$.*

It is clear that setting the leading coefficient to zero, while leaving all other coefficients unchanged will give us the minimal perturbation in this case. This additional check has to be added to algorithm C and R. For the least squares problem, we have to extend the matrices in (2) and (7) by an additional row and column of the same structure, whereas the vector of constants will become the coefficient vector of f without further additions. The sum in the denominator of the norm expressions $\mathcal{N}_m(\alpha)$ will now run from $k = 0$ to n.

We could look at the monic case as just one special instance of *linear equality constraints* imposed on the coefficients of the perturbed polynomial, and extend our algorithms to this respect as well. This more general method could also be used to preserve sparsity in the given polynomials. Inequality constraints on the other hand can lead to cases where no minimal solution exists in the least square sense (see our example above). Figure 2 is a (simplified)

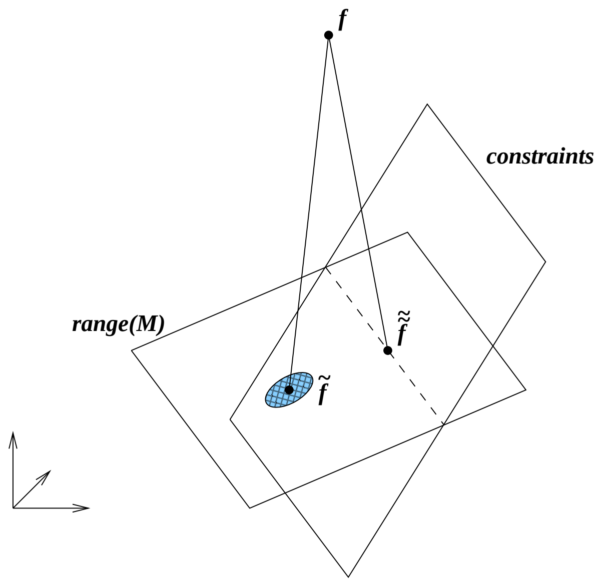

Figure 2:

illustration for some of the possible cases involving constraints on the coefficients of $\tilde{f}$ (here shown for dimension 3). The linear constraints can define a hyperplane (or a lower-dimensional varity) in coefficient space, in which case we have to project the coefficient vector of f onto the intersection of the range of the matrix $\mathbf{P}$ and that hyperplane to obtain the coefficients of the "constrained" perturbed polynomial $\tilde{\tilde{f}}$. Inequality constraints (similar to the one on the leading coefficient in example 3) on the other hand could exclude the optimal solution (indicated by the shaded area in the picture).

7 Conclusions and Future Directions

The method presented in the previous sections results in polynomial-time algorithms for computing nearest unstable polynomials in the l^2-norm for a variety of stability domains. The reduced problem of finding the nearest perturbed polynomial with a given root can also be solved in the l^∞ and l^1-norm. For the general case, finding the *parametric* minimum, in the l^∞-norm in particular, seems to be an open problem at current time. An encouraging result can be obtained by Stiefel's geometric method in [22] for the following restricted problem:

Problem 3 *Let $f \in \mathbb{R}[x]$, monic and $\alpha \in \mathbb{R}$. Find $\tilde{f} \in \mathbb{R}[x]$, monic such that $\tilde{f}(\alpha) = 0$ and $\delta_\infty = \|f - \tilde{f}\|_\infty$ is minimal.*

Because of the special structure of (2), one can derive an explicit formula for the symbolic minimum in the l^∞-sense (see also [9]):

$$\delta_\infty = \left| \frac{f(\alpha)}{\sum_{k=0}^{n-1} |\alpha^k|} \right| \qquad (10)$$

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Correction made February 7, 2010 on page 240, Section 6.2, formula for $\mathcal{N}_m(\alpha)$: w_k^{-1} replaces w_k in the denominator; this correction also applies to the formula in [12].

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8 Appendix

8.1 Proof of Theorem 2

First, we note that the column vectors of $\mathbf{P}$ form a basis for an $(n-1)$-dimensional subspace of $\mathbb{C}^n$. Therefore, the unitary complement of the column-space is one-dimensional. It can easily be verified that

$$\mathbf{w} = [1, \bar{\alpha}, \bar{\alpha}^2, \ldots, \bar{\alpha}^{n-1}]^{tr}$$

is orthogonal to any column vector of $\mathbf{P}$, thus constituting a normal vector to the hyperplane spanned by the columns of $\mathbf{P}$. Here, and in the following, we use the standard definition for the inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$:

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{y}^* \mathbf{x} = \sum_{k=1}^{n} \mathbf{x}_k \bar{\mathbf{y}}_k \,.$$

The vector of the minimal perturbations δ is the orthogonal projection of $\mathbf{b} = [a_0, \ldots, a_{n-2}, a_{n-1} + \alpha]^{tr}$ onto $\mathbf{w}$:

$$\delta = \frac{\mathbf{b} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w} = \frac{f(\alpha)}{\sum_{k=0}^{n-1} (\bar{\alpha}\alpha)^k} \mathbf{w} \,,$$

which is the vector $[\delta_0, \ldots, \delta_{n-1}]$ in (4). The derivation of (3), $\mathcal{N}_m(\alpha) = \|\delta\|^2 = \delta \cdot \delta$ is obvious.

8.2 Proof of Theorem 3

For Hurwitz polynomials, we could state (6) as approximate GCD problem for the two polynomials g and h (this is also true for the case of complex coefficients, where $f(\mathbf{i}t) = g(t) + \mathbf{i}h(t)$ and $g, h \in \mathbb{R}[t]$), and use the techniques from [11] and [12] to derive the minimum norm change and the perturbations. However, their methods also require both g and h to be monic, which condition is, in general, violated by one of the polynomials. Additionally, the derivation presented here, will give us more insights into the geometric properties of this minimization problem.

For the real Hurwitz case, the column vectors of the matrix $\mathbf{R}$ in (7) form a basis for a $(n-2)$-dimensional subspace of $\mathbb{R}^n$. It is easy to verify that the two vectors

$$\begin{aligned} \mathbf{v}_1 &= [1, 0, -t^2, 0, t^4, 0, \ldots] \,, \quad \text{and} \\ \mathbf{v}_2 &= [0, 1, 0, -t^2, 0, t^4, \ldots] \end{aligned}$$

span the orthogonal complement of the column-space of $\mathbf{R}$. The vectors

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} \,, \quad \text{and} \quad \mathbf{u}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$$

constitute an orthonormal basis for this 2-dimensional subspace. Therefore, the matrix

$$\mathbf{V}\mathbf{V}^{tr} \,, \quad \text{where} \quad \mathbf{V} = [\mathbf{u}_1 \mid \mathbf{u}_2]$$

is a *projector* matrix in the sense of [8], sec. 2.4. As a linear map, it projects any vector from $\mathbb{R}^n$ onto the subspace spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. In our case, we have to project the vector $\mathbf{c} = [a_0, \ldots, a_{n-2} - t^2, a_{n-1}]$, i.e., we have to compute $\mathbf{V}\mathbf{V}^{tr}\mathbf{c}$ in order to obtain δ. As one can see from the special structure of $\mathbf{R}$, $\mathbf{v}_1$ and $\mathbf{v}_2$, as well as the matrix $\mathbf{V}\mathbf{V}^{tr}$, odd and even rows stay separated, which allows us to compute δ_j for odd and even numbers in terms of the coefficients of h and g, respectively. We also note, that odd rows in $\mathbf{V}\mathbf{V}^{tr}$ are scaled by $1/\|\mathbf{v}_1\|^2$, even rows by $1/\|\mathbf{v}_2\|^2$, resulting in the denominators of the formulas in (8) and (9).

8.3 Approximate Solutions to Inconsistent Systems

Here, we describe how to compute approximate solutions to over-determined linear systems of real equations

$$\sum_{j=1}^{n} a_{k,j} x_j = b_k \,, \quad \text{for} \quad 1 \le k \le m \,, \quad \text{where} \quad m \ge n \,,$$

in the l^∞ and l^1-sense, by re-formulating the problem as a linear program. We follow [6], section 2.3; an early reference for the l^∞-case is [23].

An approximate l^∞-solution $\mathbf{x} \in \mathbb{R}^n$ has to minimize

$$\max_{1 \le k \le m} |\delta_k| \,, \quad \text{where} \quad \delta_k = \sum_{j=1}^{n} a_{k,j} x_j - b_k \,.$$

We introduce an additional free variable w, and formulate the problem as a linear program with $2m$ constraints:

$$\begin{array}{lll} \text{Minimize:} & w & \\ \text{Subject to:} & w \geq \delta_k & 1 \leq k \leq m \\ & w \geq -\delta_k & 1 \leq k \leq m \end{array} \qquad (11)$$

Similarly, in the l^1-norm, we have to find $\mathbf{x} \in \mathbb{R}^n$ that minimizes

$$\sum_{k=1}^{m} |\delta_k| \ , \quad \text{where} \quad \delta_k = \sum_{j=1}^{n} a_{k,j} x_j - b_k \, .$$

With $2m$ additional variables d_k^+ and d_k^-, as well as $3m$ constraints, we obtain the linear program:

$$\begin{array}{lll} \text{Minimize:} & \sum_{k=1}^{m} \left(d_k^+ + d_k^- \right) & \\ \text{Subject to:} & d_k^+ - d_k^- = \delta_k & 1 \leq k \leq m \\ & d_k^+ \geq 0 & 1 \leq k \leq m \\ & d_k^- \geq 0 & 1 \leq k \leq m \end{array} \qquad (12)$$

As noted in [6], either d_k^+ or d_k^- (or both) will be equal to zero for each k. The minimal solution to the linear program (11) will automatically satisfy that condition.