J. Symbolic Computation (1997) **subm.**, 000–000

The Kharitonov theorem and its applications in symbolic mathematical computation†

MARKUS A. HITZ
ERICH KALTOFEN

Rensselaer Polytechnic Institute, Troy, NY 12180
North Carolina State University, Raleigh, NC 27606

(Received 19 May 1997)

The Kharitonov theorem provides a means of performing sensitivity analysis for the complex roots of polynomials whose coefficients (in power base) are perturbed. In particular, it gives a computationally feasible algorithm for testing if the roots remain contained on the left hand side of the Gaussian plane if one perturbes each coefficient of a monic polynomial by a given amount. We survey an abstract approach that leads to generalizations from the literature and our own, which imposes containment of the roots within a circular sector centered in the origin of the Gaussian plane.

1. Introduction

For the problem of computing complex roots of polynomials, it is well known that the location of the roots may be very sensitive to coefficient perturbations (Wilkinson 1964). In this paper we deal with the problem of diagnosing if a polynomial has such behaviour. In the past, various results deriving bounds of root displacement of a complex polynomial from the size of perturbations of the coefficients were published (e.g., the ones cited in (Marden 1966)). Conversely, theorems elucidating the relationship between coefficient perturbations and root locations are rare, or lead to impractical algorithms in terms of computational costs. One of the few exceptions is the seminal result by V. L. Kharitonov (1978a). He showed that for interval polynomials with real coefficients, it is necessary and sufficient to test just *four* special members of the polynomial family in order to decide that all polynomials have their roots in the left half of the Gaussian plane (i.e., that they are *Hurwitz*). He extended his result to complex coefficients in a follow-up paper; eight test polynomials are required in this case.

Sensitivity analysis is an important methodolgy for dealing with symbolic/numeric problem formulations. The inputs are given with imprecise, i.e., floating point coefficients and the algorithms must decide whether within a given perturbation of the coefficients problem instances exist that satisfy the wanted properties. A classical problem is the perturbation of the coefficients to make inconsistent system of linear equations solvable.

† This material is based on work supported in part by the National Science Foundation under Grants No. CCR-9319776.

0747–7171/90/000000 + 00 $03.00/0

In fact, the method of least squares finds the smallest such perturbation of the constant right side coefficient vector, and total least squares methods solve the general problem, all measuring distance in the l_2 norm, i.e., the standard Euclidean norm. The Eckart & Young Theorem finds the distance of a non-singular input matrix to the nearest singular matrix (see Demmel (1987)). The singular value decomposition of a matrix generalizes this result to compute the distance to the nearest matrix of a given rank. Karmarkar and Lakshman Y. N. (1996) computed for two polynomials with complex coefficients the closest pair of polynomials, again measured in l_2 norm of the combined coefficient vector, that share a common root. The Kharitonov theorem yields a contribution of such a problem formulation in the area of complex root finding.

Despite of having been published in a Western journal (Kharitonov 1979), the theorem remained widely unknown until 1983, when it was introduced to the control theory community by Barmish and Bialas. Especially in the late eighties it sparked increased research activities in the area of robust control. The significance of Kharitonov's theorem to computer algebra has not been recognized so far. We summarize some of the results, and give a general procedure to extend Kharitonov's method to other domains and norms. There is an interesting interdependency between root domain, norm, and the set of test polynomials, which is not fully explored yet. Some general conditions for the existence of Kharitonov-like test sets for complex domains have been derived. However, for most cases, exponentially many members are required. Evidently, for test sets of finite cardinality, we immediately obtain a polynomial time decision procedure for root clustering in a given domain of a whole family of polynomials. As the result of Tempo (1989) illustrates, it can be benefical to allow more general (weighted) norms instead of restricting ourselves to l_p norms.

After some notational prelimnaries in section 2, we introduce the *zero exclusion principle* (appealing to a simple topological argument) in section 3. The original proof of Kharitonov's theorem was based on the Hermite-Biehler theorem. In section 4, we present a proof using geometric arguments, basically following Minnicelli et al. (1989). The proof becomes just one particular instance of a more general procedure for deriving test sets in a generic way, only making use of the zero exclusion principle. Another such example (for robust Schur stability) will be discussed in section 5, together with selected results on the complexity of more general settings. We point out the problems and limitations of interval polynomials. Finally, in section 6, we extend the result by Tempo (1989), giving further evidence that more general norms can be useful for extending the range of applicable stability domains. We conclude with a brief outlook on further areas of research.

2. Preliminaries

We are looking at families, i.e., sets $\mathcal{F}$ of univariate polynomials over the rational or complex numbers. The individual polynomials are denoted by

$$f(z, \mathbf{a}) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\,, \quad \text{where} \quad \mathbf{a} = (a_0, \ldots, a_{n-1})^{tr}\,.$$

The families collect those $f(z, \mathbf{a})$ whose coefficient vectors $\mathbf{a}$ are the values of continuous functions or are defined by constraints. We restrict ourselves to monic polynomials, i.e. the highest order coefficient will always be 1. It also means that all members of the family have the same degree n, thus avoiding any "degree-drop" problem.

DEFINITION 1. *We distinguish the following domains:*

coefficient space	A	*the subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ which is contains all the coefficient vectors* $\mathbf{a}$.
root space	$\mathcal{R}(A)$	*the set of all (complex) roots of an entire family of polynomials* $\mathcal{F} = \{f(z,\mathbf{a}) \mid \mathbf{a} \in A\}$.
image space	$\mathcal{F}(z_0)$	*the image of A under the maps $f \in \mathcal{F}$ evaluated at $z = z_0$*

For the following, let $\mathcal{D} \subset \mathbb{C}$ be an open domain whose boundary $\partial\mathcal{D}$ has a piecewise smooth (C^2), and positively oriented parametrization $t \mapsto \omega$, $t \in I_{\mathcal{D}}$, where $I_{\mathcal{D}}$ is a real interval. Either interval boundary may be at infinity.

We adopt the following definition from control theory:

DEFINITION 2. *Given a simply connected domain $\mathcal{D} \subset \mathbb{C}$, the family $\mathcal{F}$ of polynomials $f(z,\mathbf{a})$ is called $\mathcal{D}$-**stable** if $\mathcal{R}(A) \subset \mathcal{D}$.*

Special cases are:	Hurwitz stability	if	$\mathcal{D} = \{z \mid \Re(z) < 0\}$ is the left half-plane
	Schur stability	if	$\mathcal{D} = \{z \mid \lvert z\rvert < 1\}$ is the open unit disk.

An important property of $\mathcal{D}$-stable polynomials is stated in the following lemma:

LEMMA 1. *Let f monic, and $\mathcal{D} \subset \mathbb{C}$ a simply connected,* **convex** *domain, where the contour $\partial\mathcal{D}$ is a positively oriented Jordan curve with parametrization $t \mapsto \omega(t)$, as defined above. Then $\arg(f(\omega(t)))$ is a continuous and strictly increasing function of t.*

PROOF. The property is a direct consequence of the argument principle. See also Marden (1966) for a geometric interpretation. □

For real coefficients given by intervals, and $\mathcal{D}$ being the left half-plane, we can now state the original version of Kharitonov's theorem as follows:

THEOREM 1. *(Kharitonov 1978a)*
Let $f(z,\mathbf{a}) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ with $\mathbf{a} \in \mathbb{R}^n$. The family of polynomials ("interval polynomial")

$$\mathcal{F} \;=\; \{f(z,\mathbf{a}) \mid \underline{a}_k \le a_k \le \overline{a}_k,\; 0 \le k < n\}$$

is Hurwitz if and only if the four vertex polynomials

$$\begin{aligned} k_{11} &= g_1 + h_1 \qquad & k_{12} &= g_1 + h_2 \\ k_{21} &= g_2 + h_1 \qquad & k_{22} &= g_2 + h_2 \end{aligned}$$

are Hurwitz, where

$$\begin{aligned} g_1(z) &= \underline{a}_0 + \overline{a}_2 z^2 + \underline{a}_4 z^4 + \cdots \\ g_2(z) &= \overline{a}_0 + \underline{a}_2 z^2 + \overline{a}_4 z^4 + \cdots \\ \\ h_1(z) &= \underline{a}_1 z + \overline{a}_3 z^3 + \underline{a}_5 z^5 + \cdots \\ h_2(z) &= \overline{a}_1 z + \underline{a}_3 z^3 + \overline{a}_5 z^5 + \cdots \end{aligned} \tag{2.1}$$

The generalization to the case of complex coefficients and eight test polynomials is straight-forward (see Kharitonov (1978b) and Minnichelli *et al.* (1989)). It actually uses two sets of polynomials, four for the upper half-plane, and four for the lower half-plane respectively. A proof of theorem 1 will be presented in section 4. Clearly, any algorithm for testing individual polynomials for the Hurwitz property yields an efficient algorithm for the entire interval family. Algorithms based on Sturm sequences can be found in (Gantmacher 1959). Modern algorithms are based on approximately computing the integral along the imaginary axis (Schönhage 1982, Schönhage *et al.* 1994).

3. The Zero Exclusion Principle

Theorems regarding the root locations of polynomials are usually based one way or another on the argument principle of complex variables (see Marden (1966) and Gantmacher (1959)). However, they do not easily carry over to statements for families of polynomials. Another fundamental principle is more suitable for that purpose, and its (implicit) use in robust stability analysis goes back at least to (Frazer and Duncan 1929).

THEOREM 2. *(Zero Exclusion Principle)*
Let $\mathcal{F}$ be defined by a connected set $A \subset \mathbb{R}^n$ (or $A \subset \mathbb{C}^n$). If $f(z, \tilde{\mathbf{a}})$ is $\mathcal{D}$-stable for some $\tilde{\mathbf{a}} \in A$, and $0 \notin f(\omega(t), A)$ for all $t \in I_{\mathcal{D}}$, then $\mathcal{F}$ is $\mathcal{D}$-stable.

PROOF. (see, e.g., (Rantzer 1992))
For $0 \le k \le n$, let $A_k = \{\mathbf{a} \in A \,|\, f(z, \mathbf{a}) \text{ has exactly } k \text{ zeros in } \mathcal{D}\}$. Then $A = \bigcup_{k=0}^{n} A_k$, where each A_k is open by the continuity of the map from coefficient space onto root space and the premise that $\mathcal{D}$ is open. Topologically, the connected set A cannot be the union of disjoint *open* nonempty sets. Given the assumption that A_n contains at least one element, the sets $A_0, \ldots, A_{n-1}$ have to be empty, and thus $A = A_n$. □

From the proof, one can see immediately that the criterium applies not only to determination of root clustering but also to *root separation*. Under the different assumption that there is some $\tilde{\mathbf{a}} \in A$ for which $f(z, \tilde{\mathbf{a}})$ has exactly m roots in $\mathcal{D}$ $(0 \le m \le n)$, we end up with $A = A_m$. The rather strong condition that $0 \notin f(\omega(t), A)$ for all $t \in I_{\mathcal{D}}$ prevents zeroes from crossing $\partial\mathcal{D}$ into and out of $\mathcal{D}$ when walking along $\partial\mathcal{D}$. Continuity is the crucial property, and since $f(\omega(t), a)$ also depends continously on t, the condition for t can be replaced by the two requirements:

$$0 \notin f(\omega(\tilde{t}), A) \;\text{ for some }\; \tilde{t} \in I_{\mathcal{D}}\,, \quad \text{and} \quad 0 \notin \partial f(\omega(t), A) \;\text{ for all }\; t \in I_{\mathcal{D}} \setminus \{\tilde{t}\}\,.$$

This second formulation is computationally more favorable, if there is an easy way of expressing $\partial f(\omega(t), A)$. In general, however, $f(\omega(t), A)$ has a highly non-trivial shape, and the test implied by the theorem becomes intractable. Instead of applying the procedure to the whole family $\mathcal{F}$, one tries to select a (finite) subset $\mathcal{T}$ of *test polynomials*, given by a subset $A_{\mathcal{T}} \subset A$. Test sets usually span polytopes or rectangular n-dimensional boxes in coefficient space. Necessity for $\mathcal{D}$-stability is obvious, since $\mathcal{T} \subset \mathcal{F}$. To make the test a sufficient criterium, one has to guarantee that $f(\omega(t), \mathbf{a})$ is contained in the space spanned by the test polynomials for all $t \in I_{\mathcal{D}}$ and for all $\mathbf{a} \in A$. Usually, this is the more challenging part. Although the image space $\mathcal{F}(\omega(t))$ does, in general, not cover the entire convex hull $H(t)$ of the image space of the test polynomials, because of continuity, zero would have to pass through the boundary of the convex hull prior to entering it.

More precisely, if $0 \in \mathcal{F}(\omega(t))$ for some particular $t \in I_{\mathcal{D}}$, there has to be some $\hat{t} \in I_{\mathcal{D}}$, such that $0 \in \partial H(\hat{t})$. We now can outline a *general decision procedure* for $\mathcal{D}$-stability of a whole family $\mathcal{F}$ of polynomials by testing the subset $\mathcal{T} \subset \mathcal{F}$:

Containment Determine the convex hull $H(t)$ of the image $f(\omega(t), A_{\mathcal{T}})$ at $t \in I_{\mathcal{D}}$. Show for any $t \in I_{\mathcal{D}}$ that $f(\omega(t), A) \subset H(t)$.

"Basis" Find some $\tilde{t} \in I_{\mathcal{D}}$ (or at infinity), such that $0 \notin H(\tilde{t})$.

"Induction" Show that 0 cannot enter $H(t)$ for all the other $t \in I_{\mathcal{D}} \setminus \{\tilde{t}\}$.

4. Proof of the Theorem

As a first example for the application of the zero exclusion principle, we show a proof for Kharitonov's theorem following Minnichelli *et al.* (1989). First, we have to define the border of the domain $\mathcal{D}$, in this case the border of the left half-plane. Although $\mathcal{D}$ is unbounded, we can think of the root space being contained in a half-circle around the origin of the complex plane (see figure 1). As the radius of the half-circle approaches

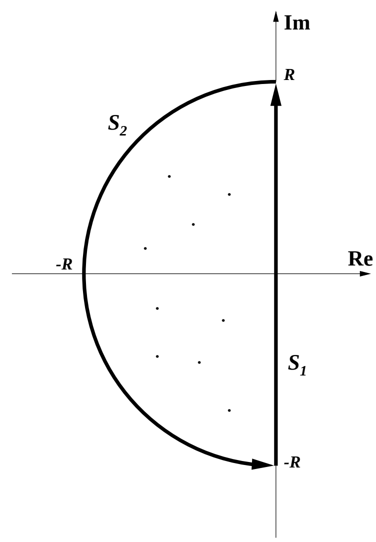

Figure 1. Domain Boundary

infinity, the segment S_1 becomes the imaginary axis, and S_2 reduces to a jump from $+\infty\mathbf{i}$ to $-\infty\mathbf{i}$. Note that all polynomials in $\mathcal{F}$ have constant degree n. Therefore, there are no roots at inifinity, and zero cannot enter the domain by crossing S_2. For real coefficients, all complex roots appear in pairs of conjugates, i.e. they are located symmetrically around the real axis, whereas the real roots lie on the axis itself. Thus the boundary, we have to check for zero crossings, is further reduced to the non-negative part of the imaginary axis. Its parametrization is given by the map:

$$t \mapsto \omega(t) = \mathbf{i}t\,, \quad t \in I_{\mathcal{D}} = [0, +\infty) \subset \mathbb{R}\,.$$

Referring to (2.1), we see now why the test polynomials were chosen that way: g_1 and g_2 contains only even powers of z, therefore on the domain boundary, both polynomials obtain only real values. On the other hand, h_1 and h_2 yield pure imaginary values if we substitute $\mathbf{i}t$ for z. It can easily be seen that for any $f \in \mathcal{F}$ and $t \geq 0$ the following inequalities hold:

$$\begin{array}{ccccc} \Re(g_1(\mathbf{i}t)) & \leq & \Re(f(\mathbf{i}t)) & \leq & \Re(g_2(\mathbf{i}t)), \quad \text{and} \\ \Im(h_1(\mathbf{i}t)) & \leq & \Im(f(\mathbf{i}t)) & \leq & \Im(h_2(\mathbf{i}t)). \end{array}$$

Consequently, $f(\mathbf{i}t)$ is contained in a rectangle, the convex hull $H(t)$ of the four vertex polymomials k_{11}, k_{12}, k_{21}, and k_{22} at $t \in [0, +\infty)$. This geometric interpretation is historically attributed to Dasgupta (1988) (see figure 2).

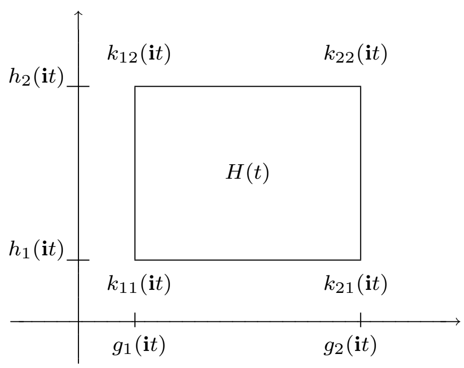

Figure 2. Dasgupta's Rectangle (1988)

This concludes the *containment* step. We now have to find some $\tilde{t} \in [0, +\infty)$ such that $0 \notin H(\tilde{t})$. We could make an argument for $+\infty$, but for the special case of real coefficients, the following Lemma shows that we can also choose $\tilde{t} = 0$:

Lemma 2. *Let the polynomial f be monic with real coefficients. If f is Hurwitz then all of its coefficients are positive.*

Proof. The real parts of all roots of f are negative. Expressing the coefficients in terms of the roots, and taking into account that complex roots appear as conjugate pairs, shows immediately that they have to be positive. □

Finally, we have to prove that zero cannot enter $H(t)$ for $t > 0$. The corners of $H(t)$ cannot pass through the origin, because the test polynomials are Hurwitz. Therefore zero would have to enter through one of the edges. Without loss of generality, we may assume that the origin is on the bottom edge for some $\hat{t} \in [0, +\infty)$. Lemma 1 states that $\arg(f(\mathbf{i}t))$ is a strictly increasing function of t, for f Hurwitz. Increasing t by an infinitesimal amount $\partial t > 0$, would place $k_{11}(\hat{t} + \partial t)$ below the real axis, and $k_{21}(\hat{t} + \partial t)$ above, which would inevitably tilt the rectangle (see figure 3). However, the construction of the test polynomials does not permit this to happen.

This concludes the proof of Kharitonov's theorem for the case of real coefficients. We would like to add a few remarks:

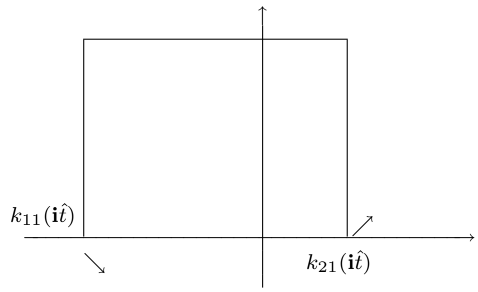

Figure 3. Tilting

1. The proof in Minnichelli *et al.* (1989) does not explicitly appeal to the zero exclusion principle, and therefore requires some additional intermediate steps.
2. The *tilting* argument used in the final step can be applied to other areas of constant shape (e.g., "diamonds"). An example is given in the next section.
3. The "box" $H(t)$ moves along the path given by the (unperturbed) center polynomial of the family, as t goes from 0 to $+\infty$. Size and aspect ratio of its sides change, but the shape stays rectangular, with edges parallel to the axes (see figure 4).

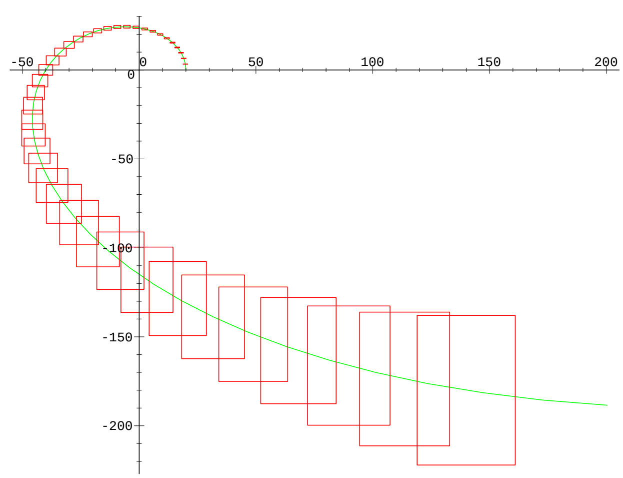

Figure 4. Moving Box

5. Generalizations

Looking at Kharitonov's theorem, the question of generalizing the the result comes immediately to mind. There are three possible directions for investigations:

1. other domains $\mathcal{D}$ than the left half-plane.
2. other norms than the infinity norm used to define A.
3. test sets of different shape.

Unfortunately, there is a strong interdependency among these areas of generalization. For interval polynomials, Rantzer was able to prove that the corner polynomials $f(z, \mathbf{b})$ where $b_i \in \{\underline{a}_i, \overline{a}_i\}$ form a test set only under a fairly restrictive condition for the domain. The following definition is used in this setting.

DEFINITION 3. *Given an open domain $\mathcal{D} \subset \mathbb{C}$. $\mathcal{D}$ is called* **weakly Kharitonov** *if for any interval polynomial (over the real or complex numbers) of degree n, it suffices to test the 2^n corner polynomials to establish $\mathcal{D}$-stability.*

Although such a test set would not lead to computationally tractable algorithms, it is helpful in establishing lower bound criteria, like the following:

THEOREM 3. *(Rantzer 1992)*
Given a domain $\mathcal{D} \subset \mathbb{C}$. $\mathcal{D}$ is weakly Kharitonov if and only if $\mathcal{D}$ and the "inverse" of $\mathcal{D}$, $1/\mathcal{D} \equiv \{z|\, zd = 1,$ for some $d \in \mathcal{D}\}$ are **convex**.

This result immediately excludes a considerable number of interesting domains, among others the open unit disc. By moving away from interval polynomials to families defined by other norms, some positive results can be achieved (see below). There exists also a definition for a domain to be *strongly Kharitonov*, which requires only a finite selection (which is still allowed to depend on the degree n) of the corner polynomials to be tested. However, so far there are only a few special cases (besides the left half-plane) known to have the strong Kharitonov property (e.g., "left sectors", see Foo and Soh (1989)).

The most general result, the so-called *edge theorem* by Bartlett *et al.* (1988), gives an upper bound for testing a polytope of polynomials for $\mathcal{D}$-stability:

THEOREM 4. *Let the family of monic polynomials be defined by $A = \mathrm{span}(\mathbf{a}^{(1)}, \ldots, \mathbf{a}^{(n)})$, where $\mathbf{a}^{(i)} \in \mathbb{R}^n$ (or $\mathbf{a}^{(i)} \in \mathbb{C}$) for $1 \leq i \leq n$. Further, let $\mathcal{D} \subset \mathbb{C}$ be a simply connected domain. Then, $\mathcal{R}(A) \subset \mathcal{D}$ iff the root space of all "exposed" edges of A is contained in $\mathcal{D}$.*

Exposed edges are the linear combination of pairs $(\mathbf{a}^{(i)}, \mathbf{a}^{(j)})$ of coefficient vectors:

$$\mathbf{a} = (1 - \lambda)\mathbf{a}^{(i)} + \lambda \mathbf{a}^{(j)}, \quad 0 \leq \lambda \leq 1,$$

such that $\mathbf{a}$ lies on the contour of the convex hull of A. Obviously, there are combinatorically many exposed edges; e.g., if A is a hypercube, their number is $n2^{n-1}$.

More encouraging are attempts to use other norms, and test sets of different shape. For Schur-stability (when $\mathcal{D}$ is the open unit disc), Tempo (1989) was able to establish a result which is equivalent to Kharitonov's theorem in the sense that it also uses four test polynomials. However, they are arranged in a tilted square ("diamond"), and the polynomial family is defined by constraints given in a slightly modified l_1 norm rather than the infinity norm:

THEOREM 5. *(Tempo 1989) Let $\mathcal{F}$ a family of complex polynomials be given by a "center" polynomial $f(z, a^*)$ and some precision $\epsilon \in \mathbb{R}^+$, such that every $f(z, a) \in \mathcal{F}$ satisfies the condition:*

$$\frac{1}{2}|\alpha_0 - \alpha_0^*| + \frac{1}{2}|\beta_0 - \beta_0^*| + \sum_{k=1}^{n-1} |\alpha_k - \alpha_k^*| + \sum_{k=1}^{n-1} |\beta_k - \beta_k^*| \leq \epsilon\,,$$

where $a_k = \alpha_k + \mathbf{i}\beta_k$ and $a_k^* = \alpha_k^* + \mathbf{i}\beta_k^*$ for $0 \leq k < n$. Further assume that $\alpha_0 \neq 0$ and $\beta_0 \neq 0$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is Schur stable iff the four polynomials:

$$\begin{array}{ll} f(z, a^*) + 2\epsilon\,, & f(z, a^*) + 2\mathbf{i}\epsilon\,, \\ f(z, a^*) - 2\epsilon\,, & f(z, a^*) - 2\mathbf{i}\epsilon \end{array}$$

are Schur stable.

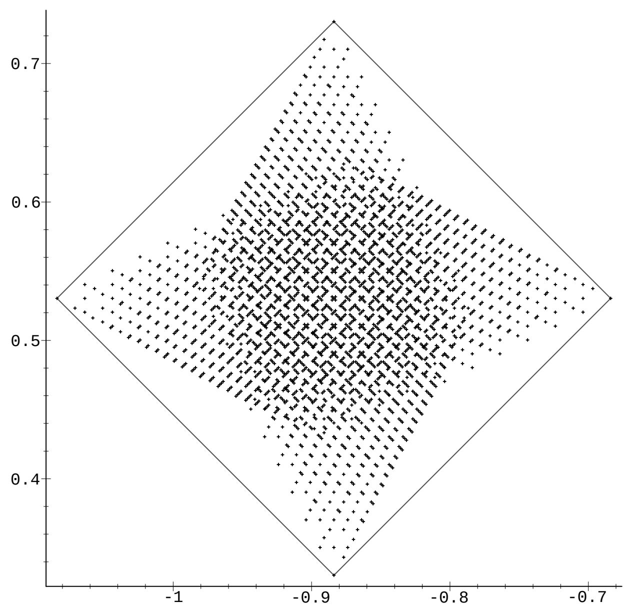

Figure 5. Function values contained in diamond

The convex hull $H(t)$ of the images of the test polynomials is a diamond of constant size. The factors 1/2 for the constant terms in the norm expression and, respectively, 2 in the definition of the test polynomials are essential. If we would use the regular l_1 norm, the "cloud" of function values (evaluated on the unit circle) of the family would be circular. In order to contain them, a larger area has to be chosen. By using twice the precision ϵ, the cloud is stretched into the corners of the diamond formed by the function values of the test polynomials, and stays within the diamond. A typical snapshot is shown in figure 5. We will state the inequalities assuring containment for a more general case in the next section. The condition on α_0 and β_0 guarantees that zero is not in $H(t)$ for at least one t; it could be replaced by a more sophisticated argument. To prove that zero cannot enter $H(t)$, the tilting argument can be used again, this time applied to an edge of the diamond. The requirements for Lemma 1 are fulfilled, because the unit disc is convex, and the corner polynomials are assumed to be Schur-stable.

So far, no results other than tests based on function plots have been published for other l_p norms. Regarding the shape of $H(t)$, one can always increase the number of test polynomials at the expense of computational costs to get a closer approximation to the actual image space.

Our Extension

Based on the result by Tempo (1989) from the previous section, we present here an extension to circles around the origin with given, but arbitrary radius $r > 0$, and sectors of such circles. It demonstrates that weighted norms can help to keep the test set small while still covering a wider range of stability domains. We refer to figure 6 for the parameters used in the remainder of this section. Let the polynomial family $\mathcal{F}$ be defined by the following condition for its complex coefficients:

$$\frac{1}{2}|\alpha_0 - \alpha_0^*| + \frac{1}{2}|\beta_0 - \beta_0^*| + \sum_{k=1}^{n-1} r^k |\alpha_k - \alpha_k^*| + \sum_{k=1}^{n-1} r^k |\beta_k - \beta_k^*| \le \epsilon\,,$$

where $f(z, a^*)$ is the center polynomial, and $\epsilon \in \mathbb{R}^+$ a given precision as in theorem 5. Now, we can state

THEOREM 6. *Let $\mathcal{D}$ be an open disc or sector with radius r around the origin, and let the four test polynomials be defined as in Theorem 5. Then, $\mathcal{F}$ is $\mathcal{D}$-stable iff the test polynomials are $\mathcal{D}$-stable.*

PROOF. We only show containment in the diamond, the second part of the proof uses the tilting argument based on Lemma 1 again. It is applicable because both domains are convex.

The contour of the disc with radius r can be parametrized by

$$t \mapsto \omega(t) = re^{\mathbf{i}t}\,, \quad \text{for} \quad t \in [0, 2\pi)\,,$$

whereas a line originating at the origin with slant angle Θ and length r is given by

$$t \mapsto \omega(t) = te^{\mathbf{i}\Theta}\,, \quad \text{for} \quad t \in [0, r]\,.$$

We derive the inequalities for the circle, the ones for the line are identical, except that the roles of t and Θ, and r and t respectively, are exchanged, and taking into account that $0 \le t^k \le r^k$ for $1 \le k \le n-1$ in that case.

We define the following functions for the real and the imaginary part of $f \in \mathcal{F}$ at $z = \omega(t)$ and a particular coefficient vector $\mathbf{a}$:

$$R(t, \mathbf{a}) \equiv \Re(f(\omega(t), \mathbf{a})) = \sum_{k=0}^{n-1} r^k(\alpha_k \cos kt - \beta_k \sin kt) + r^n \cos nt\,,$$

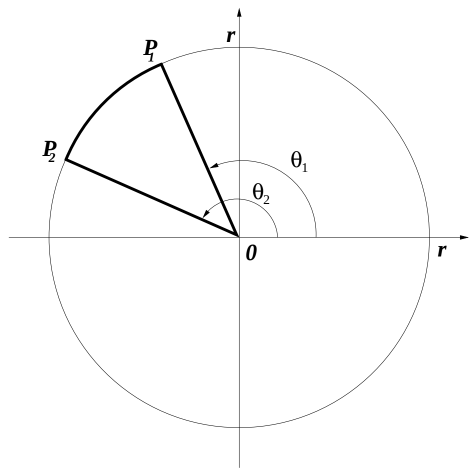

Figure 6. Disc and Sector of Radius r

and

$$I(t,\mathbf{a}) \equiv \Im(f(\omega(t),\mathbf{a})) = \sum_{k=0}^{n-1} r^k(\beta_k \cos kt + \alpha_k \sin kt) + r^n \sin nt\,.$$

The edges of the diamond are 45 degree lines, i.e. we have the condition:

$$|R(t,\mathbf{a}) - R(t,\mathbf{a}^*)| + |I(t,\mathbf{a}) - I(t,\mathbf{a}^*)| \le 2\epsilon$$

for $f(\omega(t)))$ to lie inside or on the boundary of the diamond. Expanding the left hand side yields:

$$\left|\sum_{k=0}^{n-1} r^k\left((\alpha_k - \alpha_k^*)\cos kt - (\beta_k - \beta_k^*)\sin kt\right)\right| + \left|\sum_{k=0}^{n-1} r^k\left((\beta_k - \beta_k^*)\cos kt + (\alpha_k - \alpha_k^*)\sin kt\right)\right|$$

$$\le 2\sum_{k=1}^{n-1} r^k\,|\alpha_k - \alpha_k^*| + 2\sum_{k=1}^{n-1} r^k\,|\beta_k - \beta_k^*| + |\alpha_0 - \alpha_0^*| + |\beta_0 - \beta_0^*|$$

taking into account that $|\cos kt| \le 1$ and $|\sin kt| \le 1$ for all k. The last line is of course $\le 2\epsilon$ by the norm condition for the coefficients of the family. In fact, the condition was constructed that way. □

In the case where $\mathcal{D}$ is a sector, the root locations of the test polynomials have to be checked in three steps: first whether they are contained in the circle of radius r, then whether they lie to the left of the line given by Θ_1, finally whether they lie in the right half-plane of the line defined by Θ_2. These tests can either be numerical (using an appropriate root-finder), or symbolic. In the latter case, we have to assume that the lines are not given by angles, but by rational coordinates of some point, other than the origin, lying on that line. The location of the roots relative to the domain boundary can then be determined via (generalized) Sturm sequences. There exist rational parametrizations of the circle; for the lines we have to work in an algebraic extension of the rationals (see Gantmacher (1959) and Kaltofen (1990)).

6. Conclusions and Future Directions

We presented selected results surrounding the theorem by V. L. Kharitonov, and presented a proof for his classic result based on the zero exclusion principle. Our approach tried to uncover the underlying general procedure that is common to results for various domains and norms. The most intriguing aspect of Kharitonov's theorem is that, like with Sturm sequences, a problem on an infinite set can be decided by testing only finitely many instances. It therefore seems to be especially suited for applications in symbolic computation.

Although the theoretical results for the infinity norm do not leave much room for the development of efficient algorithms for deciding stability of interval polynomials in other domains, weighted norms and different shapes for the containment provide alternatives. Another, yet unexplored approach would use computational methods, be they symbolic or numerical, for deciding that zero cannot cross the image domain boundary as the the test polynomials are evaluated along the input contour. This would further open up the way for methods that test for stable *root separation*, because for that problem the famous tilting argument seems not to apply.

References

Barmish, B. R., Kang, H. I. (1993). A survey of extreme point results for robustness of control systems. *Automatica*, 29(1):13–35.

Bartlett, A. C., Hollot, C. V., Lin, H. (1988). Root locations of an entire polytope of polynomials: it suffices to check the edges. *Math. Control Signals Systems*, 1:61–71.

Dasgupta, S. (1988). Kharitonov's theorem revisited. *Systems and Control Letters*, 11:381–384.

Demmel, J. W. (1987). On condition numbers and the distance to the nearest ill-posed problem. *Numer. Math.*, 51:251–289.

Foo, Y. K., Soh, Y. C. (1989). Root clustering of interval polynomials in the left-sector. *Systems & Control Letters*, 13:239–245.

Frazer, R. A., Duncan, W. J. (1929). On the criteria for stability for small motions. *Proceedings of the Royal Society A*, 124:642–654.

Gantmacher, F. R. (1959). *Matrix Theory*. Chelsea Publishing Company, New York, N.Y.

Kaltofen, E. (1990). Computing the irreducible real factors and components of an algebraic curve. *AAECC*, 1:135–148.

Karmarkar, N., Lakshman Y. N. (1996). Approximate polynomial greatest common divisors and nearest singular polynomials. In Lakshman Y. N., editor, *Proc. Internat. Symp. Symbolic Algebraic Comput. ISSAC '96*, pages 35–42, New York, N. Y. ACM Press.

Kharitonov, V. L. (1978a). Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differensial'nye Uravnenya*, 14(11):2086–2088.

Kharitonov, V. L. (1978b). On a generalization of a stability criterion. *Akademii nauk Kazakhskoi SSR, Seria fiziko-mathematicheskaia*, 1:53–57.

Kharitonov, V. L. (1979). Asymptotic stability of an equilibrium position of a family of systems of linear differential equations. *Differential Equations*, 14:1483–1485.

Marden, M. (1966). *Geometry of Polynomials*. Mathematical Surveys, *Number 3*. American Mathematical Society, Providence, R.I., 2^{nd} edition.

Minnichelli, R. J., Anagnost, J. J., Desoer, C. A. (1989). An elementary proof of Kharitonov's stability theorem with extensions. *IEEE Trans. Automatic Control*, 34(9):995–998.

Rantzer, A. (1992). Stability conditions for polytopes of polynomials. *IEEE Trans. Auto. Control*, 37(1):79–89.

Schönhage, A. (1982). The fundamental theorem of algebra in terms of computational complexity. Tech. report, Univ. Tübingen.

Schönhage, A., Grotefeld, A. F. W., Vetter, E. (1994). *Fast algorithms a multitape Turing machine implementation*. Bibliographisches Institut, Mannheim, Germany.

Tempo, R. (1989). A simple test for Schur stability of a diamond of complex polynomials. In *Proc. 28th Conf. Decision Control*, pages 1892–1895. IEEE.

Wilkinson, J. H. (1964). *Rounding errors in algebraic processes.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey.