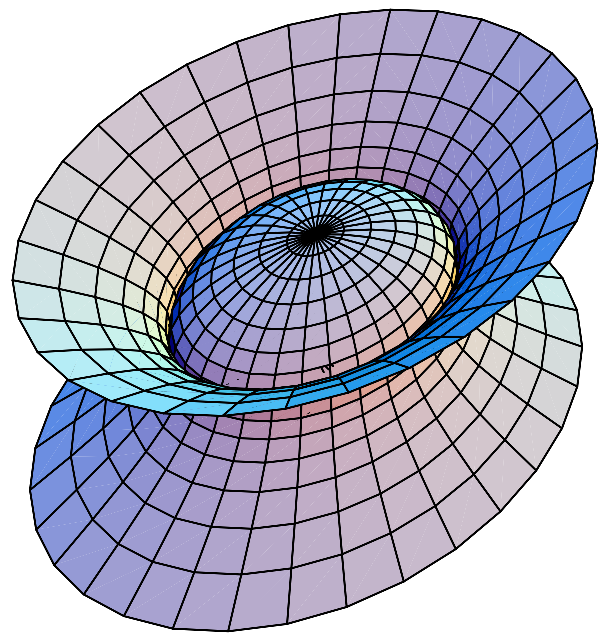
Efficient Algorithms for Computing the Nearest Polynomial With Parametrically Constrained Roots and Factors

Erich Kaltofen
North Carolina State University
www.kaltofen.net

Joint work with: Markus Hitz (North Georgia College)
Lakshman Y. N. (Bell Labs)

Factorization of nearby polynomials over the complex numbers

$$81x^4+16y^4-648z^4+72x^2y^2-648x^2-288y^2+1296=0$$



$$(9x^2+4y^2+18\sqrt{2}z^2-36)(9x^2+4y^2-18\sqrt{2}z^2-36)=0$$

$$81x^4+16y^4-648.003z^4+72x^2y^2+.002x^2z^2+.001y^2z^2$$
$$-648x^2-288y^2-.007z^2+1296=0$$

Open Problem *[Kaltofen LATIN'92]*
Given is a polynomial $f(x,y) \in \mathbb{Q}[x,y]$ *and* $\varepsilon \in \mathbb{Q}$.

Decide in polynomial time in the degree and coefficient size if there is a factorizable $\hat{f}(x,y) \in \mathbb{C}[x,y]$ *with* $\|f - \hat{f}\| \leq \varepsilon$,

for a reasonable coefficient vector norm $\|\cdot\|$.

Theorem *[Hitz, Kaltofen, Lakshman ISSAC'99]*
We can compute in polynomial time in the degree and coefficient size if there is a $\tilde{f}(x,y) \in \mathbb{C}[x,y]$ *with a factor of a* ***constant*** *degree and* $\|f - \tilde{f}\|_2 \leq \varepsilon$.

Numerical algorithms

Conclusion on my exact algorithm [JSC 1985]:

"*D. Izraelevitz at Massachusetts Institute of Technology has already implemented a version of algorithm 1 using complex floating point arithmetic. Early experiments indicate that the linear systems computed in step (L) tend to be* ***numerically ill-conditioned.*** *How to overcome this numerical problem is an important question which we will investigate.*"

Galligo and Watt [ISSAC'97]: substitute variables by generic linear forms; then certain coefficients in true factors must vanish.

Stetter, Haung, Wu and Zhi [ISSAC'2K]: Hensel lift factor combinations numerically and eliminate extraneous factors early

Univariate Problem: Given $f \in \mathbb{C}[z]$ and $\alpha \in \mathbb{C}$.
Find $\tilde{f} \in \mathbb{C}[z]$, such that

$$\tilde{f}(\alpha) = 0, \quad \text{and} \quad \|f - \tilde{f}\| = \min .$$

Let

$$\begin{aligned} f(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ \tilde{f}(z) &= (z - \alpha)\left(u_{n-1} z^{n-1} + u_{n-2} z^{n-2} + \cdots + u_0\right) \\ &= u_{n-1} z^n + (u_{n-2} - \alpha) z^{n-1} + \cdots + (u_0 - \alpha u_1) z - \alpha u_0 \end{aligned}$$

In terms of linear algebra:

$$\|f - \tilde{f}\| = \min_{\mathbf{u} \in \mathbb{C}^n} \left\| \underbrace{\begin{bmatrix} -\alpha & & & 0 \\ 1 & -\alpha & & \\ & \ddots & \ddots & \\ 0 & & 1 & -\alpha \\ & & & 1 \end{bmatrix}}_{\mathbf{P}} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}}_{\mathbf{u}} - \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix}}_{\mathbf{b}} \right\| \qquad (1)$$

(1) is an over-determined linear system of equations:

Linear program, if $\|\cdot\|$ is the $\begin{cases} \infty\text{-norm, or} \\ 1\text{-norm} \end{cases}$

Least squares problem, if $\|\cdot\|$ is the 2-norm (Euclidean).

Solutions for the 2-norm in closed form:

$$N_{min}(\alpha) = \|f - \tilde{f}\|^2 = \frac{\overline{f(\alpha)}\, f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k}, \quad f_j - \tilde{f}_j = \frac{(\overline{\alpha})^j f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k}$$

(also derived in Corless et al. [ISSAC'95] via SVD)

An ∞-norm example: $x^2+1 = 1\cdot x^2 + 0\cdot x + 1$

$$\min_{\substack{\tilde{a}_2, \tilde{a}_1, \tilde{a}_0 \in \mathbb{R} \text{ such that} \\ \exists \alpha \in \mathbb{R}\colon\ \tilde{a}_2\alpha^2 + \tilde{a}_1\alpha + \tilde{a}_0 = 0}} \Big(\max\{|1-\tilde{a}_2|, |0-\tilde{a}_1|, |1-\tilde{a}_0|\} \Big)$$

$$=?$$

Sensitivity analysis: Kharitonov [1978] theorem

Given are $2n$ rational numbers $\underline{a}_i, \overline{a}_i$.
Let P be the *interval* polynomial

$$P = \{x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid \underline{a}_i \leq a_i \leq \overline{a}_i \text{ for all } 0 \leq i < n\}.$$

Then every polynomial in P is *Hurwitz* (all roots have negative real parts), if and only if the four “corner” polynomials

$$g_k(x) + h_l(x) \in P, \quad \text{where } k = 1,2 \text{ and } l = 1,2,$$

with

$$g_1(x) = \underline{a}_0 + \overline{a}_2 x^2 + \underline{a}_4 x^4 + \cdots, \quad h_1(x) = \underline{a}_1 + \overline{a}_3 x^3 + \underline{a}_5 x^5 + \cdots,$$
$$g_2(x) = \overline{a}_0 + \underline{a}_2 x^2 + \overline{a}_4 x^4 + \cdots, \quad h_2(x) = \overline{a}_1 + \underline{a}_3 x^3 + \overline{a}_5 x^5 + \cdots$$

are Hurwitz.

Constraining a Root Locus to a Curve

Let Γ be a piecewise smooth curve with finitely many segments, each having a parametrization $\gamma_k(t)$ in a single real parameter t.

For a given polynomial $f \in \mathbb{C}[z]$, we want to find a minimally perturbed polynomial $\tilde{f} \in \mathbb{C}[z]$ that has (at least) one root on Γ.

Parametric Minimization

We substitute the parametrization $\gamma_k(t)$ for the indeterminate α in $N_{min}(\alpha)$. The resulting expression is a function in $t \in \mathbb{R}$.

It attains its minima at its *stationary* points. We have to compute the *real* roots of the derivative.

The derivative of the norm-expression is determined *symbolically*, but the roots can be computed numerically.

Example (of a monic polynomial)

$$f(z) = z^3 + (2.41 - 3.50\mathbf{i})z^2 + (2.76 - 5.84\mathbf{i})z - 1.02 - 9.25\mathbf{i}$$

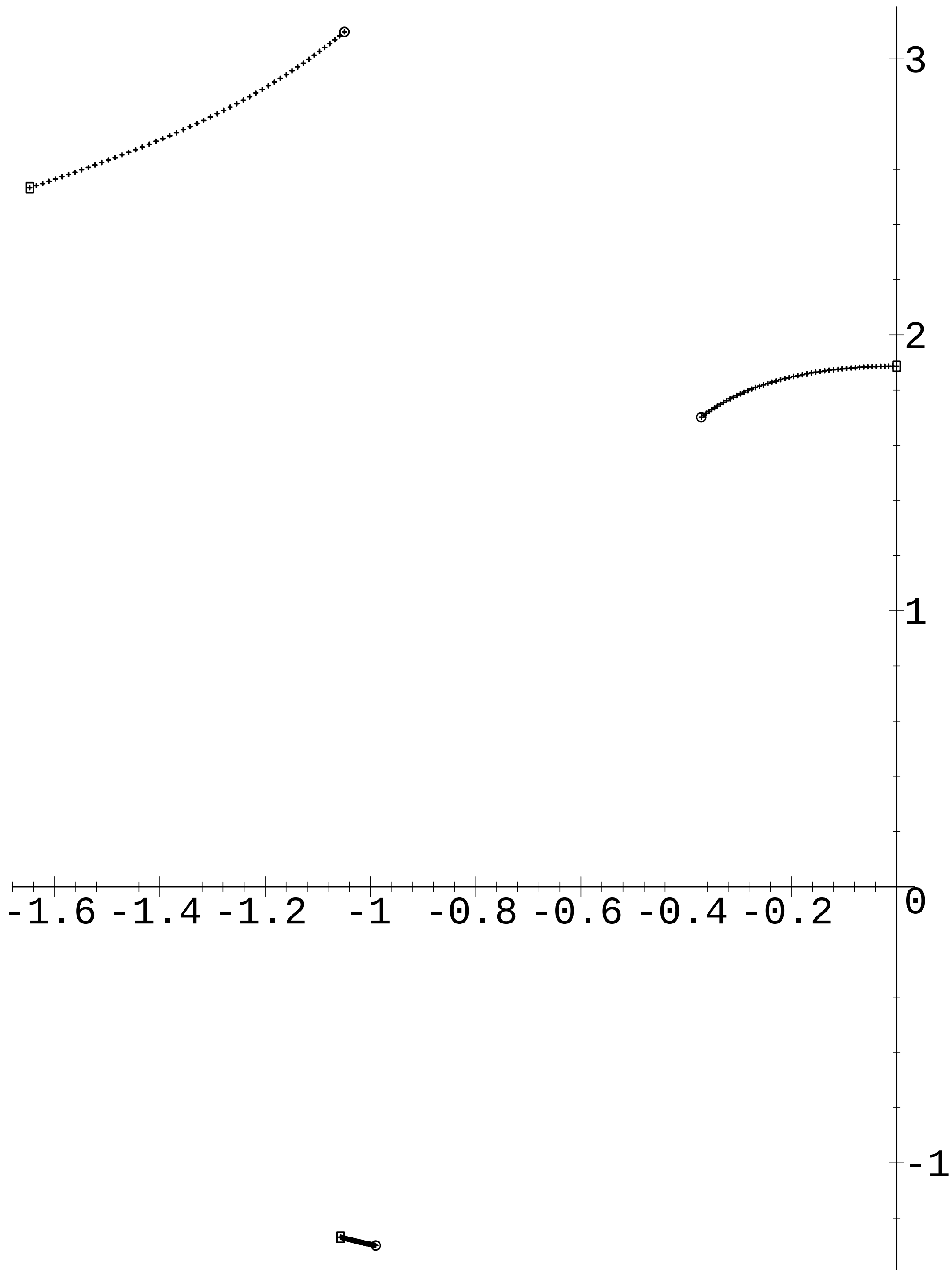
Root locations: $-1.04 + 3.10\mathbf{i}$, $-.99 - 1.30\mathbf{i}$, $-.37 + 1.70\mathbf{i}$

(satisfies Hurwitz conditions)

Nearest unstable polynomial, i.e., one root is on imaginary axis:

$$\tilde{f}(z) = z^3 + (2.7037 - 3.1492\mathbf{i})z^2 + (2.5740 - 5.6842\mathbf{i})z - 1.1026 - 9.3486\mathbf{i}.$$

Radius of stability in the 2-norm: 0.533567.


3
2
1
0
-1
-1.6 -1.4 -1.2 -1 -0.8 -0.6 -0.4 -0.2

Bivariate factorization

Given $f = \sum f_{i,j} x^i y^j \in \mathbb{C}[x, y]$ **absolute irreducible**, find

$$\tilde{f} = (c_0 + c_1 x + c_2 y) u(x, y) \in \mathbb{C}[x, y], \quad \deg(\tilde{f}) \leq \deg(f),$$

such that $\| f - \tilde{f} \|_2$ is minimal.

("nearest polynomial with a linear factor").

Approach: minimize parametric least square solution in the real and imaginary parts of the $c_i = \alpha_i + \beta_i \mathbf{i}$.

$\rightarrow$ must minimize least squares solution with 6 parameters.

$\rightarrow$ yields polynomial system with a **fixed number of variables**, hence polynomial time.

Special case: nearest polynomial with root α :

$$\delta(\alpha) = \min_{\mathbf{u}\in\mathbb{R}^n} \|\mathbf{P}\mathbf{u} - \mathbf{b}\|_\infty = \left| \frac{\sum_{i=0}^n \lambda_i a_i}{\sum_{i=0}^n |\lambda_i|} \right| = \left| \frac{f(\alpha)}{\sum_{i=0}^n |\alpha^i|} \right| . \qquad (2)$$

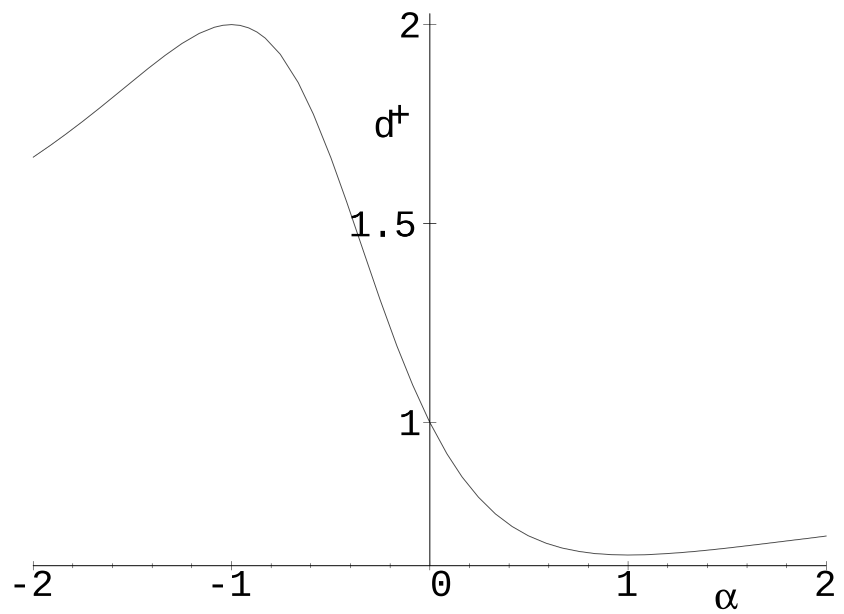
(also derived by Manocha & Demmel [1995])

Stiefel's theorem also gives algorithm for finding $\mathbf{u}$.

Parametric α : must minimize rational function (2).

Generalization to l^p-norm, where $1 \leq p \leq \infty$ (Hitz 1999):

$$\delta(\alpha) = \frac{|f(\alpha)|}{\left(\sum_{k=0}^n |\alpha^k|^q\right)^{1/q}}, \quad \frac{1}{q} + \frac{1}{p} = 1\,, \quad \text{and} \quad \frac{1}{\infty} = 0$$



$$f(x) = x^2 + 1,\ \tilde{f}(x) = \frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{3} = \frac{1}{3}(x-1)^2,\ \delta = \frac{2}{3}.$$

Sensitivity analysis: component-wise nearest singular matrix

Given are $2n^2$ rational numbers $\underline{a}_{i,j}, \overline{a}_{i,j}$.
Let A be the *interval* matrix

$$A = \left\{ \begin{bmatrix} a_{1,1} & \ldots & a_{n,n} \\ \vdots & & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix} \mid \underline{a}_{i,j} \leq a_{i,j} \leq \overline{a}_{i,j} \text{ for all } 1 \leq i,j \leq n \right\}.$$

Does A contain a singular matrix?
This problem is *NP-complete* (Poljak & Rohn 1990).