A Proof of the Monotone Column Permanent (MCP) Conjecture for Dimension 4 via Sums-Of-Squares of Rational Functions*

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ABSTRACT

For a proof of the monotone column permanent (MCP) conjecture for dimension 4 it is sufficient to show that 4 polynomials, which come from the permanents of real matrices, are nonnegative for all real values of the variables, where the degrees and the number of the variables of these polynomials are all 8. Here we apply a hybrid symbolic-numerical algorithm for certifying that these polynomials can be written as an exact fraction of two polynomial sums -of-squares (SOS) with rational coefficients.

Categories and Subject Descriptors: I.1.2 [Symbolic and Algebraic Manipulation]: Algorithms; G.1.6 [Numerical Analysis]: Global optimization

General Terms: algorithms, experimentation

Keywords: semidefinite programming, sum-of-squares, Monotone Column Permanent Conjecture, hybrid method

1. INTRODUCTION

We first state the proven conjecture, which deals with permanental characteristic polynomials of real matrices. For an $n \times n$ matrix A the permanent of A, denoted perm(A), is defined as

$$\text{perm}(A) := \sum_{\sigma \in S_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

Let E_n denote the $n \times n$ matrix of all 1's. The following conjecture was given by Haglund, Ono, Wagner [2]:

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Conjecture 1 (Monotone Column Permanent (MCP)). *Let A be a real $n \times n$ matrix whose columns are weakly increasing, i.e., $a_{i,j} \le a_{i+1,j}$ for all $1 \le i < n$ and $1 \le j \le n$. Then all of the zeros of $perm(zA + E_n) \in \mathbb{R}[z]$ are real.*

The $n = 3$ case of the MCP Conjecture has been proved by Ray Mayer [9], but it was open for $n \ge 4$. Mayer also proved that for $n = 4$, in order to prove MCP Conjecture, it is sufficient to prove the following theorem.

Theorem 1. *For all $1 \le i \le j \le 3$, the following polynomials are nonnegative*

$$\begin{aligned} p_{i,j} = & perm([\eta_{i+1}, \eta_1, u, v]) perm([\eta_{j+1}, \eta_1, u, v]) \\ & - perm([\eta_{i+1}, \eta_{j+1}, u, v]) perm([\eta_1, \eta_1, u, v]) \ge 0 \end{aligned}$$

for all real values for the variables x, y, a, b, c, d, e, f, where $u = x\eta_1 + a^2\eta_2 + b^2\eta_3 + c^2\eta_4$, $v = y\eta_1 + d^2\eta_2 + e^2\eta_3 + f^2\eta_4$, and

$$\eta_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \eta_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \quad \eta_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}, \quad \eta_4 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}.$$

All polynomials $p_{i,j}$ for $1 \le i \le j \le 3$ are of degree 8 with 8 variables. Among them, the polynomials $p_{1,1}$, $p_{3,3}$ are perfect squares. In the following section, we will show that the polynomial $p_{1,3}$ can be written as polynomial sum-of-squares (SOS) and the polynomials $p_{1,2}$, $p_{2,2}$, $p_{2,3}$ can be written as polynomial SOS divided by weighted sum of squares of variables, which proves the MCP Conjecture for $n = 4$.

2. EXACT RATIONAL FUNCTION SUM-OF-SQUARES CERTIFICATES

We present a hybrid symbolic-numeric algorithm in [7] for certifying a polynomial or rational function with rational coefficients to be non-negative for all real values of the variables by computing a representation for it as a fraction of two polynomial sum-of-squares with rational coefficients. Our new approach turns the earlier methods by Peyrl and Parrilo [10] at SNC'07 and ours [6] at ISSAC'08 both based on polynomial SOS, which do not always exist, into a universal algorithm for all inputs via Artin's theorem which states that

$$\forall \xi_1,\dots,\xi_n \in \mathbb{R}\colon p(\xi_1,\dots,\xi_n) \not< 0 \; (p \text{ is } [\textit{positive semi-}]\ \textit{definite}) \qquad (1)$$

$$\Updownarrow$$

$$\exists u_0,\dots,u_m \in K[X_1,\dots,X_n]\colon p(X_1,\dots,X_n) = \sum_{i=1}^{l} \left(\frac{u_i}{u_0}\right)^2,$$

where $K = \mathbb{R}$ or $K = \mathbb{Q}$ and $p \in K[X_1,\dots,X_n]$ or $p \in K(X_1,\dots,X_n)$.

For a chosen denominator degree δ, to prove a polynomial $p \in \mathbb{Q}[X_1,\dots,X_n]$ positive semidefinite, we can apply SDP to solve the following SOS program:

$$\left.\begin{aligned} &\inf_{W} \ \ \mathrm{Trace}(W) \\ &\text{s. t. } p(X) = \frac{m_\Delta(X)^T\, W^{[1]}\, m_\Delta(X)}{m_\delta(X)^T\, W^{[2]}\, m_\delta(X)} \\ &\qquad W = \begin{bmatrix} W^{[1]} & 0 \\ 0 & W^{[2]} \end{bmatrix}, W \succeq 0, W^T = W. \end{aligned}\right\} \qquad (2)$$

Here $m_\Delta(X), m_\delta(X)$ denote the vectors of monomial terms of degree no more than Δ and δ, respectively.

For the fixed degree δ, the SOS program (2) can be solved efficiently by algorithms in GloptiPoly [3], SOSTOOLS [11], YALMIP [8] and SeDuMi [13]. However, since we are running fixed precision SDP solvers in Matlab, we can only obtain numerical positive semidefinite matrices $W^{[1]}, W^{[2]}$ which satisfy approximately

$$p(X) \approx \frac{m_\Delta(X)^T\, W^{[1]}\, m_\Delta(X)}{m_\delta(X)^T\, W^{[2]}\, m_\delta(X)}, W^{[1]} \succeq 0\ , W^{[2]} \succeq 0. \qquad (3)$$

The polynomial p can be certified as non-negative if $\widetilde{W}^{[1]}, \widetilde{W}^{[2]}$ satisfy the following conditions exactly:

$$p(X) = \frac{m_\Delta(X)^T\, \widetilde{W}^{[1]}\, m_\Delta(X)}{m_\delta(X)^T\, \widetilde{W}^{[2]}\, m_\delta(X)}, \quad \widetilde{W}^{[1]} \succeq 0\ , \widetilde{W}^{[2]} \succeq 0. \qquad (4)$$

In [7], we start with finding a rational positive semidefinite matrix $\widetilde{W}^{[2]}$ near to $W^{[2]}$ by solving the SOS program (2), then for the fixed denominator, we use Gauss-Newton iterations to refine the matrix $W^{[1]}$. The rational positive semidefinite symmetric matrix $\widetilde{W}^{[1]}$ which satisfies (4) exactly can be computed by orthogonal projection ($\widetilde{W}^{[1]}$ is of full rank) or rational vector recovery ($\widetilde{W}^{[1]}$ is singular).

Lemma 1. *The polynomial $p_{1,3}$, which has 53 monomials, can be written as polynomial SOS with 10 polynomials.*

Let $\delta = 0$ and $W^{[2]} = [1]$, by solving the SOS program (2), we obtain a 29×29 numerical positive semidefinite matrix $W^{[1]}$. After rounding the entries of $W^{[1]}$ to integers and computing its exact $\mathrm{LDL^T}$-decomposition, we can write $p_{1,3}$ as an SOS of 10 polynomials:

$$p_{1,3} = \frac{1}{12} g_1^2 + g_2^2 + \frac{1}{2}\, g_3^2 + \frac{1}{4}\, g_4^2 + \frac{1}{4}\, g_5^2 + \frac{1}{2}\, g_6^2 + \frac{1}{2}\, g_7^2 + g_8^2 + \frac{1}{2}\, g_9^2 + g_{10}^2,$$

where

$$\begin{aligned} g_1 &= 12\,xy + 6\,yb^2 + 12\,ya^2 + 6\,xe^2 + 6\,b^2d^2 + 12\,xd^2 \\ &\quad +6\,a^2e^2 + 12\,a^2d^2, \\ g_2 &= yb^2 - xe^2 + b^2d^2 - a^2e^2, \quad g_3 = 2\,yab + 2\,abe^2 + 2\,abd^2, \\ g_4 &= 4\,yac + 2\,ace^2 + 4\,acd^2, \quad g_5 = 4\,xdf + 2\,b^2df + 4\,a^2df, \\ g_6 &= 2\,xde + 2\,b^2de + 2\,a^2de, \quad g_7 = 2\,ybc + 2\,bcd^2, \\ g_8 &= bcde, \quad g_9 = 2\,xef + 2\,a^2ef, \quad g_{10} = abef. \end{aligned}$$

For the polynomials $p_{1,2}, p_{2,2}, p_{2,3}$, if we set $\delta = 0$, the SDP solver reports that the SOS program (2) is probably infeasible. Hence it is unlikely (we have not proven this) that these polynomials are sums of squares of polynomials. Even if they were, our methods first try rational coefficients, which for polynomial sums of squares are only conjectured (Sturmfels—see [4]). Therefore, in the following, we start with setting $\delta = 1$ and show that each can be written as a rational polynomial SOS divided by a weighted sum of squares of variables.

Lemma 2. *The polynomial $p_{2,2}$ which has 67 monomials can be written as polynomial SOS with 11 polynomials divided by the polynomial $a^2 + 2b^2 + c^2$.*

Letting $m_\delta(X) = [1, e, d, c, b, a, y, x]^T$ and solving the SOS program (2), we obtain the matrix $W^{[2]}$:

$$\begin{bmatrix} 10^{-6} & -10^{-15} & -10^{-15} & 10^{-14} & 10^{-14} & 10^{-14} & 10^{-6} & 10^{-6} \\ -10^{-15} & 10^{-5} & 10^{-14} & 10^{-14} & 10^{-13} & -10^{-13} & -10^{-15} & 10^{-15} \\ -10^{-15} & 10^{-14} & 10^{-5} & -10^{-13} & -10^{-14} & 10^{-13} & 10^{-14} & 10^{-15} \\ 10^{-14} & 10^{-14} & -10^{-13} & 0.629 & -10^{-13} & 10^{-13} & -10^{-14} & 10^{-13} \\ 10^{-14} & 10^{-13} & -10^{-14} & -10^{-13} & 1.26 & 10^{-13} & 10^{-14} & -10^{-13} \\ 10^{-14} & -10^{-13} & 10^{-13} & 10^{-13} & 10^{-13} & 0.632 & 10^{-14} & -10^{-13} \\ 10^{-6} & -10^{-15} & 10^{-14} & -10^{-14} & 10^{-14} & 10^{-14} & 10^{-6} & -10^{-6} \\ 10^{-6} & 10^{-15} & 10^{-15} & 10^{-13} & -10^{-13} & -10^{-13} & -10^{-6} & 10^{-5} \end{bmatrix}$$

After converting the matrix $W^{[2]}$ to a nearby rational matrix, we obtain the polynomial $\frac{2}{3}\, a^2 + \frac{4}{3}\, b^2 + \frac{2}{3} c^2$ as the denominator. Then we compute the polynomial SOS of $(a^2 + 2\, b^2 + c^2) \cdot p_{2,2}$. The singular values of the 64×64 matrix $W^{[1]}$ obtained by the SDP solver are

$$\underbrace{302., 155.2, \dots, 8.00, 2.00}_{11}, 0.144\text{e–}1, 0.143\text{e–}2, 0.165\text{e–}3, \dots.$$

After rounding the matrix $W^{[1]}$ to an integer matrix, we obtain the integer matrix $\widetilde{W}^{[1]}$ which satisfies (4) exactly. Therefore, the polynomial $(a^2 + 2\, b^2 + c^2) \cdot p_{2,2}$ can be written as an SOS of 11 polynomials:

$$\begin{aligned} (a^2 + 2\, b^2 + c^2) \cdot p_{2,2} = & \frac{1}{48} g_1^2 + g_2^2 + \frac{1}{96}\, g_3^2 + \frac{1}{2}\, g_4^2 + \frac{1}{48}\, g_5^2 + g_6^2 \\ & + \frac{1}{8}\, g_7^2 + \frac{1}{4}\, g_8^2 + \frac{1}{4}\, g_9^2 + \frac{1}{8}\, g_{10}^2 + \frac{1}{2}\, g_{11}^2, \end{aligned}$$

where

$$\begin{aligned} g_1 &= 36\,xad^2 + 24\,xae^2 + 24\,yab^2 + 36\,ya^3 + 6\,a^3f^2 + 12\,a^3e^2 \\ &\quad +24\,a^3d^2 + 12\,xaf^2 + 6\,ac^2d^2 + 48\,xya + 12\,yac^2 \\ &\quad +12\,ab^2d^2, \\ g_2 &= xad^2 - 2\,xae^2 + 2\,yab^2 + ya^3 - 1/2\,a^3f^2 - a^3e^2 \\ &\quad +2\,a^3d^2 - xaf^2 + 3/2\,ac^2d^2 + yac^2 + 3\,ab^2d^2, \\ g_3 &= 48\,yb^3 + 24\,b^3d^2 + 96\,xyb + 24\,ybc^2 + 72\,ya^2b + 24\,xbf^2 \\ &\quad +48\,xbe^2 + 72\,xbd^2 + 12\,bc^2d^2 + 12\,a^2bf^2 + 24\,a^2be^2 \\ &\quad +48\,a^2bd^2, \\ g_4 &= 4\,yb^3 + 2\,b^3d^2 + 2\,ybc^2 + 2\,ya^2b - 2\,xbf^2 - 4\,xbe^2 \\ &\quad -2\,xbd^2 + bc^2d^2 - a^2bf^2 - 2\,a^2be^2, \\ g_5 &= 12\,yc^3 + 6\,c^3d^2 + 48\,xyc + 24\,yb^2c + 36\,ya^2c + 12\,xcf^2 \\ &\quad +24\,xce^2 + 36\,xcd^2 + 12\,b^2cd^2 + 6\,a^2cf^2 + 12\,a^2ce^2 \end{aligned}$$

$+24\,a^2cd^2,$

$$g_6 = yc^3 + 1/2\,c^3d^2 + 2\,yb^2c + ya^2c + xcf^2 - 2\,xce^2 - xcd^2 + b^2cd^2 + 1/2\,a^2cf^2 - a^2ce^2,$$

$$g_7 = 8\,xade + 4\,ac^2de + 8\,ab^2de + 8\,a^3de,$$

$$g_8 = 4\,xadf + 4\,ab^2df + 4\,a^3df,$$

$$g_9 = 4xcdf + 4a^2cdf,\; g_{10} = 8xcef + 4a^2cef,\; g_{11} = 2abcdf.$$

Lemma 3. *The polynomial $p_{1,2}$, which has 77 monomials, can be written as polynomial SOS with 27 polynomials divided by the polynomial $a^2 + 2b^2 + c^2$.*

We use the same polynomial $a^2+2\,b^2+c^2$ as the denominator and compute the polynomial SOS of $(a^2+2\,b^2+c^2)\cdot p_{1,2}$. The singular values of the 77×77 matrix $W^{[1]}$ obtained by the SDP solver are

$$\underbrace{224.88, 150.88, \ldots, .92067, .89573, .45125}_{27}, 0.68340\text{e–}4\;, \;\ldots .$$

Let us truncate the matrix $W^{[1]}$ to be of rank 27 for the given tolerance 10^{-3}, and refine it by applying structure-preserving Newton iteration to its $\mathrm{LDL}^{\mathrm{T}}$- factorization. Since $W^{[1]}$ has rank deficiency 50, the orthogonal projection cannot project $W^{[1]}$ on to the cone of symmetric positive semidefinite matrices. However, after rounding the matrix $612 \cdot W^{[1]}$ to an integer matrix and then dividing it by 612, we obtain the rational matrix $\widetilde{W}^{[1]}$ which satisfies (4) exactly. Therefore, the polynomial $p_{1,2} \cdot (a^2 + 2\,b^2 + c^2)$ can be written as an SOS of 27 polynomials.

Lemma 4. *The polynomial $p_{2,3}$, which has 45 monomials, can be written as polynomial SOS with 41 polynomials divided by the polynomial $2\,a^2 + 4\,b^2 + 3\,c^2$.*

We obtain the polynomial $a^2 + 2\,b^2 + \frac{3}{2}\,c^2$ as the denominator by converting $W^{[2]}$ to a nearby rational matrix. We compute the polynomial SOS of $p_{2,3}\cdot(2\,a^2+4\,b^2+3\,c^2)$. The singular values of the 53×53 matrix $W^{[1]}$ obtained by the SDP solver are

$$\underbrace{289.14, 199.58, \ldots, 0.97004, 0.82641}_{27}, 0.0062362\;, \ldots .$$

Let us truncate the matrix $W^{[1]}$ to be of rank 27 for the given tolerance 10^{-2}, and refine it by applying structure-preserving Newton iteration to its $\mathrm{LDL}^{\mathrm{T}}$- factorization. After rounding the matrix $979 \cdot W^{[1]}$ to an integer matrix and then dividing it by 979, we obtain the rational matrix $\widetilde{W}^{[1]}$ which satisfies (4) exactly. Therefore, we can write the polynomial $(2\,a^2+4\,b^2+3\,c^2)\cdot p_{2,3}$ as an exact SOS of 27 polynomials.

It should be noted that the denominator can be chosen in many different ways. For example, using the same denominator as for $p_{2,2}$ and $p_{1,2}$, the polynomial $(a^2+2\,b^2+c^2)\cdot p_{2,3}$ can also be written as an SOS of 27 polynomials. However, if we let $m_\delta(X) = [1, e, d, c, b, a, y, x]^T$ and solve the SOS program (2), we obtain the polynomial $a^2+2b^2+2c^2+d^2+2e^2$ as the denominator after converting $W^{[2]}$ to a nearby rational matrix. The polynomial $p_{2,3}\cdot(a^2+2b^2+2c^2+d^2+2e^2)$ can be written as an SOS of 41 polynomials.

Remark 1. Mohab Safey El Din has told us that he could also with his RAGlib Maple package for real algebraic geometry determine that the polynomials $p_{1,2}, p_{1,3}, p_{2,2}, p_{2,3}$ are non-negative. He reports that his computation took less than 5 minutes (on a laptop) and the time could be reduced to less than 2 minutes by using internal routines of RAGlib and the results in [12]. RAGlib analyzes the critical and asymptotic critical values of the polynomials and does not yield an easily verifiable certificate of non-negativity such as our SOSes.

All polynomials in Maple input representation and verification of the claimed identities via Maple expansion can be downloaded from http://www4.ncsu.edu/~kaltofen/software/mcp_conj_4/.

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3. REFERENCES

[1] Haglund, J. Further investigations involving rook polynomials with only real zeros. *European J. Combinatorics 21* (1999), 1017–1037. URL: http://www.math.upenn.edu/~jhaglund/preprints/defxy.pdf.

[2] Haglund, J., Ono, K., and Wagner, D. G. Theorems and conjectures involving rook polynomials with real roots. In *Topics in Number Theory In Honor of B. Gordon and S. Chowla*, S. D. Ahlgren, G. E. Andrews, and K. Ono, Eds. Kluwer Academic Publishers (published by Springer), 1999, pp. 207–221. URL: http://www.math.upenn.edu/~jhaglund/preprints/how.pdf.

[3] Henrion, D., and Lasserre, J. B. GloptiPoly: Global optimization over polynomials with Matlab and SeDuMi. *ACM Trans. Math. Softw. 29*, 2 (2003), 165–194.

[4] Hillar, C. Sums of polynomial squares over totally real fields are rational sums of squares. *Proc. American Math. Society 137* (2009), 921–930. URL: http://www.math.tamu.edu/~chillar/files/totallyrealsos.pdf.

[5] Jeffrey, D., Ed. *ISSAC 2008* (New York, N. Y., 2008), ACM Press.

[6] Kaltofen, E., Li, B., Yang, Z., and Zhi, L. Exact certification of global optimality of approximate factorizations via rationalizing sums-of-squares with floating point scalars. In Jeffrey [5], pp. 155–163. URL: EKbib/08/KLYZ08.pdf.

[7] Kaltofen, E., Li, B., Yang, Z., and Zhi, L. Exact certification in global polynomial optimization via sums-of-squares of rational functions with rational coefficients, Jan. 2009. Manuscript, 20 pages.

[8] Löfberg, J. YALMIP : A toolbox for modeling and optimization in MATLAB. In *Proc. IEEE CCA/ISIC/CACSD Conf.* (Taipei, Taiwan, 2004). URL: http://control.ee.ethz.ch/~joloef/yalmip.php.

[9] Mayer, R., 1999. Private communication cited in [1] and via J. Buhler, March 19, 2009.

[10] Peyrl, H., and Parrilo, P. A. A Macaulay 2 package for computing sum of squares decompositions of polynomials with rational coefficients. In *SNC'07 Proc. 2007 Internat. Workshop on Symbolic-Numeric*

Comput. (New York, N. Y., 2007), J. Verschelde and S. M. Watt, Eds., ACM Press, pp. 207–208.

[11] Prajna, S., Papachristodoulou, A., and Parrilo, P. A. SOSTOOLS: Sum of squares optimization toolbox for MATLAB. URL: http://www.cds.caltech.edu/sostools.

[12] Safey El Din, M. Computing the global optimum of a multivariate polynomial over the reals. In Jeffrey [5].

[13] Sturm, J. F. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software 11/12* (1999), 625–653.

4. APPENDIX

$$\begin{aligned}
p_{1,3} = {} & 2x^2e^2f^2 + 12y^2a^4 + 12a^4d^4 + 4a^4e^4 + 4y^2b^4 + 4b^4d^4 \\
& + a^2c^2e^4 + b^4d^2f^2 + 4x^2e^4 + 24x^2yd^2 + a^2b^2e^2f^2 \\
& + 12x^2d^4 + b^2c^2d^2e^2 + 48xya^2d^2 + 12x^2y^2 + 12x^2ye^2 \\
& + 14x^2d^2e^2 + 4x^2d^2f^2 + 24xy^2a^2 + 24xa^2d^4 + 8xa^2e^4 \\
& + 12xy^2b^2 + 12xb^2d^4 + 24ya^4d^2 + 12ya^4e^2 + 14a^4d^2e^2 \\
& + 4a^4d^2f^2 + 2a^4e^2f^2 + 14y^2a^2b^2 + 14a^2b^2d^4 + 2a^2b^2e^4 \\
& + 4y^2a^2c^2 + 4a^2c^2d^4 + 8yb^4d^2 + 24xya^2e^2 + 28xa^2d^2e^2 \\
& + 8xa^2d^2f^2 + 4xa^2e^2f^2 + 24xyb^2d^2 + 4xyb^2e^2 \\
& + 8xb^2d^2e^2 + 4xb^2d^2f^2 + 28ya^2b^2d^2 + 8ya^2b^2e^2 \\
& + 12a^2b^2d^2e^2 + 4a^2b^2d^2f^2 + 8ya^2c^2d^2 + 4ya^2c^2e^2 \\
& + 4a^2c^2d^2e^2 + 4yb^2c^2d^2 + 2b^4d^2e^2 + 2y^2b^2c^2 + 2b^2c^2d^4,
\end{aligned}$$

$$\begin{aligned}
p_{2,2} = {} & 24x^2yf^2 + 16yb^4d^2 + 40x^2d^2e^2 + 4a^4f^2e^2 + 16a^4e^2d^2 \\
& + 72a^2xy^2 + 4b^2c^2d^4 + 8a^4f^2d^2 + 4yc^4d^2 + 8xc^2d^4 \\
& + 48xy^2b^2 + 20a^2y^2c^2 + 16xb^2d^4 + 16a^4e^2y + 48x^2ye^2 \\
& + 40a^2xd^4 + 4a^2xf^4 + 20x^2d^2f^2 + 72x^2yd^2 + c^4d^4 \\
& + 40a^4d^2y + 40a^2y^2b^2 + 16x^2e^2f^2 + 8a^4f^2y + 24xy^2c^2 \\
& + 4a^4e^4 + 16a^4d^4 + 16y^2b^2c^2 + 8xyb^2f^2 + 16xyb^2e^2 \\
& + 56xyb^2d^2 + 16a^2xe^4 + 4y^2c^4 + 4b^4d^4 + 16y^2b^4 \\
& + 4x^2f^4 + 16x^2e^4 + 28x^2d^4 + 48x^2y^2 + 8xb^2d^2e^2 \\
& + 4xb^2d^2f^2 + 28xyc^2d^2 + 8xyc^2e^2 + 8xyc^2f^2 \\
& + 4xc^2d^2e^2 + 4xc^2d^2f^2 + 16yb^2c^2d^2 + 16a^2xe^2f^2 \\
& + 2a^2c^2d^2f^2 + 4a^2b^2d^2f^2 + 104a^2xyd^2 + 56a^2xye^2 \\
& + 28a^2xyf^2 + 24a^2yc^2d^2 + 48a^2yb^2d^2 + 24a^2xd^2f^2 \\
& + 8a^2yb^2e^2 + 4a^2yc^2e^2 + 48a^2xd^2e^2 + 4a^2yb^2f^2 \\
& + 8a^2b^2d^2e^2 + 8a^2c^2d^4 + 16a^2b^2d^4 + 4a^2c^2d^2e^2 \\
& + 4a^2yc^2f^2 + 28a^4y^2 + a^4f^4,
\end{aligned}$$

$$\begin{aligned}
p_{1,2} = {} & 8y^2b^2c^2 + 8a^4f^2e^2 + 2a^2c^2e^4 + 24x^2ye^2 + 48a^2xy^2 \\
& + 24xb^2d^4 + 12x^2yf^2 + 12xy^2c^2 + 24a^4e^2y \\
& + 28a^2b^2d^4 + 4a^2b^2e^4 + 28a^2y^2b^2 + 12a^4f^2y \\
& + 8x^2e^2f^2 + 16a^2xe^4 + 4yc^4d^2 + 28a^4e^2d^2 + a^2b^2f^4 \\
& + 24xy^2b^2 + 2b^4d^2f^2 + 4b^4d^2e^2 + 16yb^4d^2 + 4a^2xf^4 \\
& + 48a^2xd^4 + 14a^2y^2c^2 + 14a^2c^2d^4 + 8b^2c^2d^4 \\
& + 48x^2yd^2 + 14x^2d^2f^2 + 28x^2d^2e^2 + 48a^4d^2y \\
& + 24xyc^2d^2 + 4xyc^2e^2 + 4xyc^2f^2 + 8xc^2d^2e^2 \\
& + 4xc^2d^2f^2 + 16yb^2c^2d^2 + 4b^2c^2d^2e^2 + b^2c^2d^2f^2 \\
& + 48xyb^2d^2 + 12xc^2d^4 + c^4d^2e^2 + 2y^2c^4 + 2c^4d^4 \\
& + 8y^2b^4 + 8b^4d^4 + 24x^2y^2 + 24x^2d^4 + 8x^2e^4 + 2x^2f^4 \\
& + 8xyb^2e^2 + 4xyb^2f^2 + 16xb^2d^2e^2 + 8xb^2d^2f^2 + 2a^4f^4 \\
& + 14a^4f^2d^2 + 48a^2xye^2 + 28a^2yc^2d^2 + 12a^2b^2d^2f^2 \\
& + 24a^2xyf^2 + 28a^2xd^2f^2 + 56a^2xd^2e^2 + 4a^2b^2e^2f^2 \\
& + a^2c^2e^2f^2 + 8a^2yb^2f^2 + 56a^2yb^2d^2 + 16a^2yb^2e^2 \\
& + 96a^2xyd^2 + 4a^2c^2d^2f^2 + 8a^2yc^2e^2 + 24a^2b^2d^2e^2 \\
& + 16a^2xe^2f^2 + 12a^2c^2d^2e^2 + 4a^2yc^2f^2 + 24a^4y^2 \\
& + 24a^4d^4 + 8a^4e^4,
\end{aligned}$$

$$\begin{aligned}
p_{2,3} = {} & 8e^4x^2 + 36x^2yd^2 + 36xy^2a^2 + 2x^2d^2f^2 + 8xb^2d^4 \\
& + 24xy^2b^2 + 20xa^2d^4 + 24e^2xa^2d^2 + 28e^2xya^2 \\
& + 4e^2xb^2d^2 + 8e^2xyb^2 + 4e^2xa^2f^2 + 8e^4xa^2 + 24e^2x^2y \\
& + 4e^2x^2f^2 + 20e^2x^2d^2 + 8b^4y^2 + 2d^4b^4 + 4c^2b^2y^2 \\
& + d^4c^2b^2 + 8d^2b^4y + 4d^2c^2b^2y + 2y^2a^2c^2 + 2a^2c^2d^4 \\
& + 20y^2a^2b^2 + 8a^2b^2d^4 + 20ya^4d^2 + 8ya^4e^2 + 8a^4d^2e^2 \\
& + 2a^4d^2f^2 + a^4e^2f^2 + 14y^2a^4 + 8a^4d^4 + 2a^4e^4 \\
& + 4ya^2c^2d^2 + a^2c^2d^2e^2 + 24ya^2b^2d^2 + 4ya^2b^2e^2 \\
& + 4a^2b^2d^2e^2 + a^2b^2d^2f^2 + 4xa^2d^2f^2 + 24x^2y^2 \\
& + 14x^2d^4 + 52xya^2d^2 + 28xyb^2d^2.
\end{aligned}$$

$$\begin{aligned}
& p_{1,2} \cdot (a^2 + 2b^2 + c^2) = \tfrac{1}{24} g_1^2 + \tfrac{8989056}{31613255} g_2^2 + \tfrac{967365603}{4164709648} g_3^2 \\
& + \tfrac{199125180045}{27103023406} g_4^2 + \tfrac{1843005591608}{36701015987} g_5^2 + \tfrac{1}{48} g_6^2 + g_7^2 + \tfrac{70227}{172616} g_8^2 \\
& + \tfrac{1}{24} g_9^2 + 2 g_{10}^2 + \tfrac{2247264}{3316517} g_{11}^2 + \tfrac{37145}{36857} g_{12}^2 + \tfrac{306}{2185} g_{13}^2 + \tfrac{1}{4} g_{14}^2 \\
& + \tfrac{1}{8} g_{15}^2 + \tfrac{1}{4} g_{16}^2 + \tfrac{1}{2} g_{17}^2 + \tfrac{1}{4} g_{18}^2 + \tfrac{1498176}{1367855} g_{19}^2 + \tfrac{1}{2} g_{20}^2 \\
& + \tfrac{749088}{1367855} g_{21}^2 + g_{22}^2 + \tfrac{51}{137} g_{23}^2 + \tfrac{6987}{6986} g_{24}^2 + \tfrac{1}{4} g_{25}^2 + \tfrac{166464}{161423} g_{26}^2 \\
& + \tfrac{102}{337} g_{27}^2,
\end{aligned}$$

where

$$\begin{aligned}
g_1 = {} & \tfrac{2776}{153} yab^2 + 6xaf^2 + 12xae^2 + \tfrac{3851}{612} yac^2 + \tfrac{179}{612} ac^2e^2 \\
& + \tfrac{470}{153} ab^2f^2 + 24xya + \tfrac{3851}{612} ac^2d^2 + \tfrac{940}{153} ab^2e^2 + \tfrac{2776}{153} ab^2d^2 \\
& + 24xad^2 + 24a^3d^2 + 12a^3e^2 + 6a^3f^2 + 24ya^3, \\
g_2 = {} & \tfrac{211525}{140454} yab^2 + \tfrac{2473}{2448} xaf^2 - \tfrac{2473}{1224} xae^2 + \tfrac{31613255}{8989056} yac^2 \\
& + \tfrac{22532399}{8989056} ac^2e^2 + \tfrac{1981207}{1123632} ab^2f^2 + \tfrac{31613255}{8989056} ac^2d^2 \\
& - \tfrac{289007}{561816} ab^2e^2 + \tfrac{211525}{140454} ab^2d^2 - \tfrac{2473}{1224} a^3e^2 + \tfrac{2473}{2448} a^3f^2, \\
g_3 = {} & \tfrac{4164709648}{967365603} yab^2 - \tfrac{544934080}{967365603} xaf^2 + \tfrac{3817360}{6322651} xae^2 \\
& + \tfrac{1908680}{6322651} ac^2e^2 + \tfrac{1537420744}{967365603} ab^2f^2 + \tfrac{4748765728}{967365603} ab^2e^2 \\
& + \tfrac{4164709648}{967365603} ab^2d^2 + \tfrac{3817360}{6322651} a^3e^2 - \tfrac{544934080}{967365603} a^3f^2, \\
g_4 = {} & \tfrac{27103023406}{199125180045} xaf^2 + \tfrac{41979009047}{132750120030} xae^2 + \tfrac{41979009047}{265500240060} ac^2e^2 \\
& + \tfrac{27103023406}{199125180045} ab^2f^2 + \tfrac{41979009047}{132750120030} ab^2e^2 + \tfrac{41979009047}{132750120030} a^3e^2 \\
& + \tfrac{27103023406}{199125180045} a^3f^2, \\
g_5 = {} & \tfrac{36701015987}{1843005591608} xae^2 + \tfrac{36701015987}{3686011183216} ac^2e^2 + \tfrac{36701015987}{1843005591608} ab^2e^2 \\
& + \tfrac{36701015987}{1843005591608} a^3e^2, \\
g_6 = {} & \tfrac{6404}{153} ya^2b + 12bc^2d^2 + \tfrac{6404}{153} a^2bd^2 + 48xyb + 12ybc^2 \\
& + 12xbf^2 + \tfrac{2732}{153} a^2be^2 + 48xbd^2 + \tfrac{1366}{153} a^2bf^2 \\
& + 24xbe^2 + 24b^3d^2 + 24yb^3, \\
g_7 = {} & \tfrac{133}{153} ya^2b + bc^2d^2 + \tfrac{133}{153} a^2bd^2 + ybc^2 - xbf^2 - \tfrac{173}{153} a^2be^2
\end{aligned}$$

$$-\tfrac{173}{306}\,a^2bf^2-2\,xbe^2+2\,b^3d^2+2\,yb^3,$$

$$g_8=\tfrac{172616}{70227}\,ya^2b+\tfrac{172616}{70227}\,a^2bd^2+\tfrac{172616}{70227}\,a^2be^2+\tfrac{86308}{70227}\,a^2bf^2,$$

$$g_9=6\,a^2cf^2+6\,xcf^2+\tfrac{14509}{612}\,a^2cd^2+12\,b^2cd^2+24\,xyc$$
$$+24\,xcd^2+\tfrac{7165}{612}\,a^2ce^2+12\,xce^2+12\,yb^2c+\tfrac{14509}{612}\,ya^2c$$
$$+6\,c^3d^2+6\,yc^3,$$

$$g_{10}=\tfrac{1}{2}\,a^2cf^2+\tfrac{1}{2}\,xcf^2-\tfrac{1249}{2448}\,a^2cd^2+b^2cd^2-\tfrac{3697}{2448}\,a^2ce^2$$
$$-xce^2+yb^2c-\tfrac{1249}{2448}\,ya^2c+\tfrac{1}{2}\,c^3d^2+\tfrac{1}{2}\,yc^3,$$

$$g_{11}=\tfrac{3316517}{2247264}\,a^2cd^2+\tfrac{3316517}{2247264}\,a^2ce^2+\tfrac{3316517}{2247264}\,ya^2c,$$

$$g_{12}=\tfrac{36857}{37145}\,abcf^2,$$

$$g_{13}=\tfrac{2185}{306}\,yabc-\tfrac{4}{17}\,abcf^2+\tfrac{2185}{306}\,abce^2+\tfrac{2185}{306}\,abcd^2,$$

$$g_{14}=4\,a^3de+4\,xade+2\,ac^2de+4\,ab^2de,$$

$$g_{15}=8\,xbde+4\,bc^2de+8\,a^2bde+8\,b^3de,$$

$$g_{16}=2\,c^3de+4\,xcde+4\,b^2cde+4\,a^2cde,$$

$$g_{17}=2\,a^3df+2\,xadf+2\,ab^2df,$$

$$g_{18}=4\,xbdf-\tfrac{361}{612}\,bc^2df+4\,a^2bdf+4\,b^3df,$$

$$g_{19}=\tfrac{1367855}{1498176}\,bc^2df,\quad g_{20}=2\,xcdf+\tfrac{1585}{612}\,b^2cdf+2\,a^2cdf,$$

$$g_{21}=\tfrac{1367855}{749088}\,b^2cdf,\quad g_{22}=abcdf,$$

$$g_{23}=\tfrac{137}{51}\,xaef+\tfrac{1}{51}\,ac^2ef+\tfrac{137}{51}\,ab^2ef+\tfrac{137}{51}\,a^3ef,$$

$$g_{24}=\tfrac{6986}{6987}\,ac^2ef,\quad g_{25}=4\,xcef+\tfrac{745}{204}\,a^2cef,$$

$$g_{26}=\tfrac{161423}{166464}\,a^2cef,\quad g_{27}=\tfrac{337}{102}\,abcef.$$

$$p_{2,3}\cdot(2\,a^2+4\,b^2+3\,c^2)=\tfrac{1}{48}\,g_1^2+\tfrac{95052}{544391}\,g_2^2+\tfrac{65871311}{252030896}\,g_3^2$$
$$+\tfrac{1}{96}\,g_4^2+\tfrac{5750646}{61362527}\,g_5^2+\tfrac{61362527}{413399200}\,g_6^2+\tfrac{1}{72}\,g_7^2+\tfrac{8625969}{111147400}\,g_8^2$$
$$+\tfrac{13601663075}{58580229898}\,g_9^2+\tfrac{28675022535071}{131253826953026}\,g_{10}^2+\tfrac{5840795299409657}{6260864028901248}\,g_{11}^2$$
$$+\tfrac{15362468}{26871567}\,g_{12}^2+\tfrac{979}{15692}\,g_{13}^2+\tfrac{1}{8}\,g_{14}^2+\tfrac{1916882}{5732957}\,g_{15}^2+\tfrac{89}{704}\,g_{16}^2$$
$$+\tfrac{7581376}{14622527}\,g_{17}^2+\tfrac{979}{9258}\,g_{18}^2+\tfrac{1}{4}\,g_{19}^2+\tfrac{31684}{125895}\,g_{20}^2+\tfrac{1}{8}\,g_{21}^2$$
$$+\tfrac{63368}{125895}\,g_{22}^2+\tfrac{1}{6}\,g_{23}^2+\tfrac{1}{3}\,g_{24}^2+\tfrac{1}{8}\,g_{25}^2+\tfrac{1}{16}\,g_{26}^2+\tfrac{1}{12}\,g_{27}^2,$$

where

$$g_1=\tfrac{322}{89}\,yac^2+24\,yab^2+24\,xae^2+36\,xad^2+12\,ab^2d^2$$
$$+\tfrac{322}{89}\,ac^2d^2+48\,xya+36\,ya^3+12\,a^3e^2+24\,a^3d^2,$$

$$g_2=\tfrac{544391}{95052}\,yac^2+\tfrac{977}{979}\,yab^2-\tfrac{977}{979}\,xae^2+\tfrac{977}{1958}\,xad^2+\tfrac{2931}{1958}\,ab^2d^2$$
$$+\tfrac{544391}{95052}\,ac^2d^2+\tfrac{977}{1958}\,ya^3-\tfrac{977}{1958}\,a^3e^2+\tfrac{977}{979}\,a^3d^2,$$

$$g_3=\tfrac{252030896}{65871311}\,yab^2-\tfrac{252030896}{65871311}\,xae^2+\tfrac{126015448}{65871311}\,xad^2$$
$$+\tfrac{378046344}{65871311}\,ab^2d^2+\tfrac{126015448}{65871311}\,ya^3-\tfrac{126015448}{65871311}\,a^3e^2$$
$$+\tfrac{252030896}{65871311}\,a^3d^2,$$

$$g_4=96\,xyb+\tfrac{11060}{979}\,ybc^2+72\,ya^2b+48\,xbe^2+72\,xbd^2$$
$$+\tfrac{5530}{979}\,bc^2d^2+24\,a^2be^2+48\,a^2bd^2+48\,yb^3+24\,b^3d^2,$$

$$g_5=\tfrac{61362527}{5750646}\,ybc^2+\tfrac{1797}{979}\,ya^2b-\tfrac{3594}{979}\,xbe^2-\tfrac{1797}{979}\,xbd^2$$
$$+\tfrac{61362527}{11501292}\,bc^2d^2-\tfrac{1797}{979}\,a^2be^2+\tfrac{3594}{979}\,yb^3+\tfrac{1797}{979}\,b^3d^2,$$

$$g_6=\tfrac{206699600}{61362527}\,ya^2b-\tfrac{413399200}{61362527}\,xbe^2-\tfrac{206699600}{61362527}\,xbd^2$$
$$-\tfrac{206699600}{61362527}\,a^2be^2+\tfrac{413399200}{61362527}\,yb^3+\tfrac{206699600}{61362527}\,b^3d^2,$$

$$g_7=72\,xyc+\tfrac{24184}{979}\,yb^2c+\tfrac{4484}{89}\,ya^2c+36\,xce^2+54\,xcd^2$$
$$+\tfrac{12092}{979}\,b^2cd^2+18\,a^2ce^2+\tfrac{2882}{89}\,a^2cd^2,$$

$$g_8=\tfrac{111147400}{8625969}\,yb^2c-\tfrac{325816}{784179}\,ya^2c-\tfrac{2280}{979}\,xce^2-\tfrac{1140}{979}\,xcd^2$$
$$+\tfrac{55573700}{8625969}\,b^2cd^2-\tfrac{1140}{979}\,a^2ce^2-\tfrac{1238956}{784179}\,a^2cd^2,$$

$$g_9=\tfrac{58580229898}{13601663075}\,ya^2c-\tfrac{5650509232}{2720332615}\,xce^2+\tfrac{2620967984}{2720332615}\,xcd^2$$
$$-\tfrac{2825254616}{2720332615}\,a^2ce^2+\tfrac{71685069818}{13601663075}\,a^2cd^2,$$

$$g_{10}=\tfrac{131253826953026}{28675022535071}\,xce^2-\tfrac{20096916369215}{28675022535071}\,xcd^2$$
$$+\tfrac{65626913476513}{28675022535071}\,a^2ce^2-\tfrac{20096916369215}{28675022535071}\,a^2cd^2,$$

$$g_{11}=\tfrac{6260864028901248}{5840795299409657}\,xcd^2+\tfrac{6260864028901248}{5840795299409657}\,a^2cd^2,$$

$$g_{12}=\tfrac{26871567}{15362468}\,abcd^2,\quad g_{13}=\tfrac{15692}{979}\,yabc+\tfrac{13729}{979}\,abcd^2,$$

$$g_{14}=8\,xade+\tfrac{266}{979}\,ac^2de+8\,ab^2de+8\,a^3de,$$

$$g_{15}=\tfrac{5732957}{1916882}\,ac^2de,\quad g_{16}=\tfrac{704}{89}\,xcde+\tfrac{8479}{979}\,a^2cde,$$

$$g_{17}=\tfrac{14622527}{7581376}\,a^2cde,\quad g_{18}=\tfrac{9258}{979}\,abcde,$$

$$g_{19}=4\,xadf+\tfrac{29}{89}\,ab^2df+4\,a^3df,\quad g_{20}=\tfrac{125895}{31684}\,ab^2df,$$

$$g_{21}=8\,xbdf+\tfrac{683}{89}\,a^2bdf,\quad g_{22}=\tfrac{125895}{63368}\,a^2bdf,$$

$$g_{23}=6\,xcdf+6\,a^2cdf,\quad g_{24}=3\,abcdf,$$

$$g_{25}=8\,xaef+4\,a^3ef,\quad g_{26}=16\,xbef+8\,a^2bef,$$

$$g_{27}=12\,xcef+6\,a^2cef.$$