The Seven Dwarfs of Symbolic Computation and the Discovery of Reduced Symbolic Models

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`google->kaltofen`

Outline

The 7 Dwarfs of Symbolic Computation

Discovery of sparse rational function models

Solving overdetermined linear systems optimally

Berkeley 13 Dwarfs

http://view.eecs.berkeley.edu/wiki/Dwarf_Mine

“A dwarf is an **algorithmic method** that captures **a pattern** of computation and communication”

1. Dense Linear Algebra
2. Sparse Linear Algebra
3. Spectral Methods
4. N-Body Methods
5. Structured Grids
6. Unstructured Grids
7. MapReduce
8. Combinational Logic
9. Graph Traversal
10. Dynamic Programming
11. Backtrack and Branch-and-Bound
12. Graphical Models
13. Finite State Machines

How about Logic Programming, Symbolic Computation?

My 7 Dwarfs of Symbolic Computation

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Model Discovery and Verification: 1+2+3+4+5

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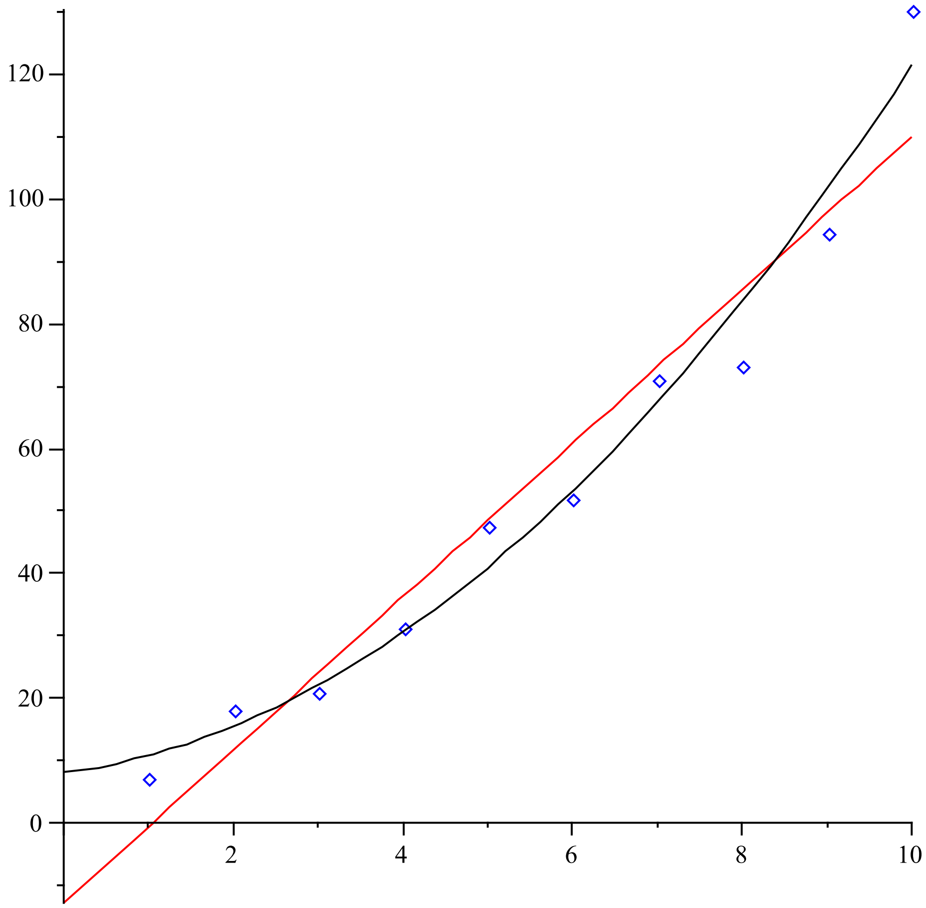
Model Discovery Example



Linear or quadratic best fit? What if best model is $\frac{2.5x^7y^{10}+1.3}{x^2-y^9}$?

Sparse rational function models [Kaltofen, Yang, Zhi '07]

By sampling black box, compute sparse representation

$$\frac{\sum_{j=1}^{t_f} \tilde{a}_j x_1^{d_{j,1}} \cdots x_n^{d_{j,n}}}{\sum_{k=1}^{t_g} \tilde{b}_k x_1^{e_{k,1}} \cdots x_n^{e_{k,n}}} = \frac{\tilde{f}}{\tilde{g}}, \quad \tilde{a}_j \neq 0, \tilde{b}_k \neq 0$$

Note: Terms are **not** known.

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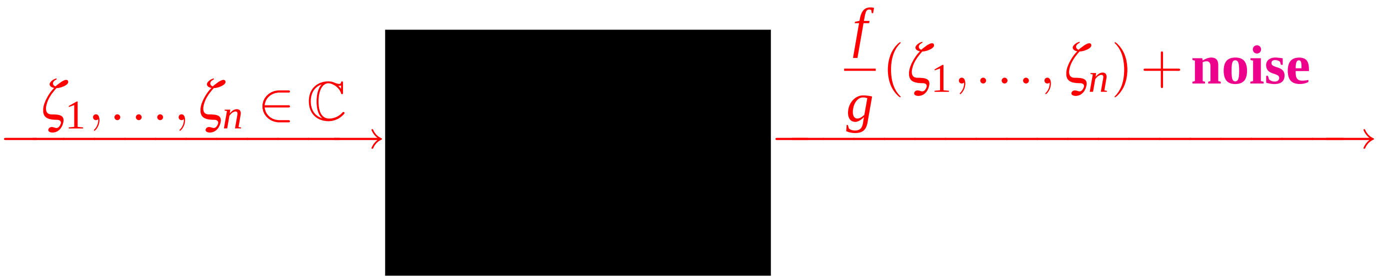

$f, g \in \mathbb{C}[x_1, \ldots, x_n], \mathrm{GCD}(f, g) = 1$

By sampling black box, compute sparse representation

$$\frac{\sum_{j=1}^{t_f} \tilde{a}_j x_1^{d_{j,1}} \cdots x_n^{d_{j,n}}}{\sum_{k=1}^{t_g} \tilde{b}_k x_1^{e_{k,1}} \cdots x_n^{e_{k,n}}} = \frac{\tilde{f}}{\tilde{g}}, \quad \tilde{a}_j \neq 0, \tilde{b}_k \neq 0$$

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ZNIPR algorithm: Sparsity by a **numeric** Zippel-Schwartz lemma
Numeric noise in values by condition
numbers in **random structured matrices**

Zippel Exact and Numerical Interpolation of Rational Functions

Idea:

- Variable by variable interpolation
  Term supports like [Kaltofen 1986], instead of [Zippel 1979]

- Early termination [cf. Kaltofen and Lee 2003]

- Structured Total Least Norm (STLN) method employed in numerical case [cf. Kaltofen, Yang, Zhi 2005, 2006]
  - STLN method to decide the support of the numerator and denominator.
  - STLN method to compute the coefficients corresponding to the support.

Ingredients of ZNIPR algorithm

Probabilistic analysis of exact method

Numerical Zippel/Schwartz lemma

Condition numbers of randomized matrices

Example (Exact Case)

Given the black box of the rational function f/g

$$f = x_1^3 + 3x_1x_2^2, \quad g = 2x_1^3 + 3x_2,$$

and the degree bounds $\bar{d} = 4$, $\bar{e} = 4$. Supppose

$$f_1 = f(x_1, a) = b_1x_1^3 + b_2x_1, \quad g_1 = g(x_1, a) = b_3x_1^3 + b_4.$$

and $\bar{D}_2 = \{1, x_2^2\}$, $\bar{E}_2 = \{1, x_2\}$, where $b_1, b_2, b_3, b_4 \in \mathsf{K} \setminus \{0\}$.

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The sets of the possible terms of f and g are

$$D_2 = \{x_1, x_1^3, x_1x_2^2\}, \quad E_2 = \{1, x_1^3, x_1^3x_2, x_2\}.$$

Example (Exact Case)

f and g can be represented as

$$f = y_1 x_1 + y_2 x_1^3 + y_3 x_1 x_2^2, \quad g = z_1 + z_2 x_1^3 + z_3 x_1^3 x_2 + z_4 x_2.$$

Pick random points $p_1, p_2 \in \mathsf{K}$ and compute the values:

$$\gamma_l = \frac{f(p_1^l, p_2^l)}{g(p_1^l, p_2^l)} \in \mathsf{K} \setminus \{0, \infty\},\ l = 0, 1, \ldots, L-1.$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & \gamma_0 & \gamma_0 & \gamma_0 & \gamma_0 \\ p_1 & p_1^3 & p_1 p_2^2 & \gamma_1 & \gamma_1 p_1^3 & \gamma_1 p_1^3 p_2 & \gamma_1 p_2 \\ p_1^2 & (p_1^3)^2 & (p_1 p_2^2)^2 & \gamma_2 & \gamma_2 (p_1^3)^2 & \gamma_2 (p_1^3 p_2)^2 & \gamma_2 p_2^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_1^{L-1} & (p_1^3)^{L-1} & (p_1 p_2^2)^{L-1} & \gamma_{L-1} & \gamma_{L-1} (p_1^3)^{L-1} & \gamma_{L-1} (p_1^3 p_2^2)^{L-1} & \gamma_{L-1} p_2^{L-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Exact Probabilistic Analysis

- Rank deficiency of G is 1 if $L \geq |D_i| \cdot |E_i|$ w.h.p.
- Using smaller L: Increment L by 1 until rank deficiency of G is 1.

Table of Exact Interpolation

Ex.	*Coeff. Range*	d_f, d_g	n	t_f, t_g	mod	*N(ZEIPR)*	*N(KY'07)*
1	[−10,10]	3,3	2	6,6	503	50	221
2	[−10,10]	5,2	4	6,3	1009	97	339
3	[−20,20]	2,4	6	2,5	120011	126	357
4	[−20,20]	1,6	8	4,8	8009	270	777
5	[−30,30]	10,5	10	7,4	4001	526	2246
6	[−10,10]	15,15	15	15,15	50021	2164	17120
7	[−10,10]	20,20	20	20,20	50021	3842	38682
8	[−30,30]	30,15	5	20,10	10007	1183	12896
9	[−50,50]	50,50	50	50,50	1000003	30405	603638
10	[−10,10]	2,8	90	10,50	1000003	10101	75082

Numerical Zippel/Schwartz Lemma

Let

$$0 \neq \Delta(\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}[\boldsymbol{i}][\alpha_1, \ldots, \alpha_s], \quad \boldsymbol{i} = \sqrt{-1},$$

$\zeta_j = \exp(\frac{2\pi \boldsymbol{i}}{P_j}) \in \mathbb{C}$, $P_j \in \mathbb{Z}_{\geq 3}$ *distinct prime numbers* $\forall 1 \leq j \leq s$
[cf. Giesbrecht, Labahn, Lee 2007]

Suppose $\Delta(\zeta_1, \ldots, \zeta_s) \neq 0$ *(use algebraic lemma to enforce)*

Then for random integers R_j *with* $1 \leq R_j < P_j$

$$\textit{Expected value}\{ \quad \left|\Delta(\zeta_1^{R_1}, \ldots, \zeta_s^{R_s})\right| \quad \} \quad \geq \quad 1.$$

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Useful for support pruning.
Also need estimates on matrix condition numbers?

Condition numbers of random $n \times n$ matrices

Entries from standard Gaussian distribution:

$$\text{Expected value}\{\log \kappa_2(G_n)\} < \log(n) + 2.258$$

[Edelman 1988; Chen and Dongarra 2005]

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Random discrete noise:

$$\kappa_2(A + R_n) = n^{O(1)} \text{ with high probability}$$

[Spielman and Teng 2004; Tao and Vu 2007]

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"... one rarely encounters ill-conditioned matrices in practice"

Approximate Polynomial GCD: Sylvester matrices

$$\begin{bmatrix} a_m & a_{m-1} & \cdots\cdots & a_0 & & & \\ & a_m & \cdots\cdots & a_1 & a_0 & \mathbf{0} & \\ & & \ddots & & \ddots & \ddots & \\ & \mathbf{0} & & a_m & a_{m-1} & \cdots\cdots & a_0 \\ b_n & b_{n-1} & \cdots\cdots & b_0 & & & \\ & b_n & \cdots\cdots & b_1 & b_0 & \mathbf{0} & \\ & & \ddots & & \ddots & \ddots & \\ & \mathbf{0} & & b_n & \cdots\cdots & \cdots\cdots & b_0 \end{bmatrix}$$

approximate polynomial GCD (common root)

for $a_m x^m + \cdots + a_0$ and $b_n x^n + \cdots + b_0$

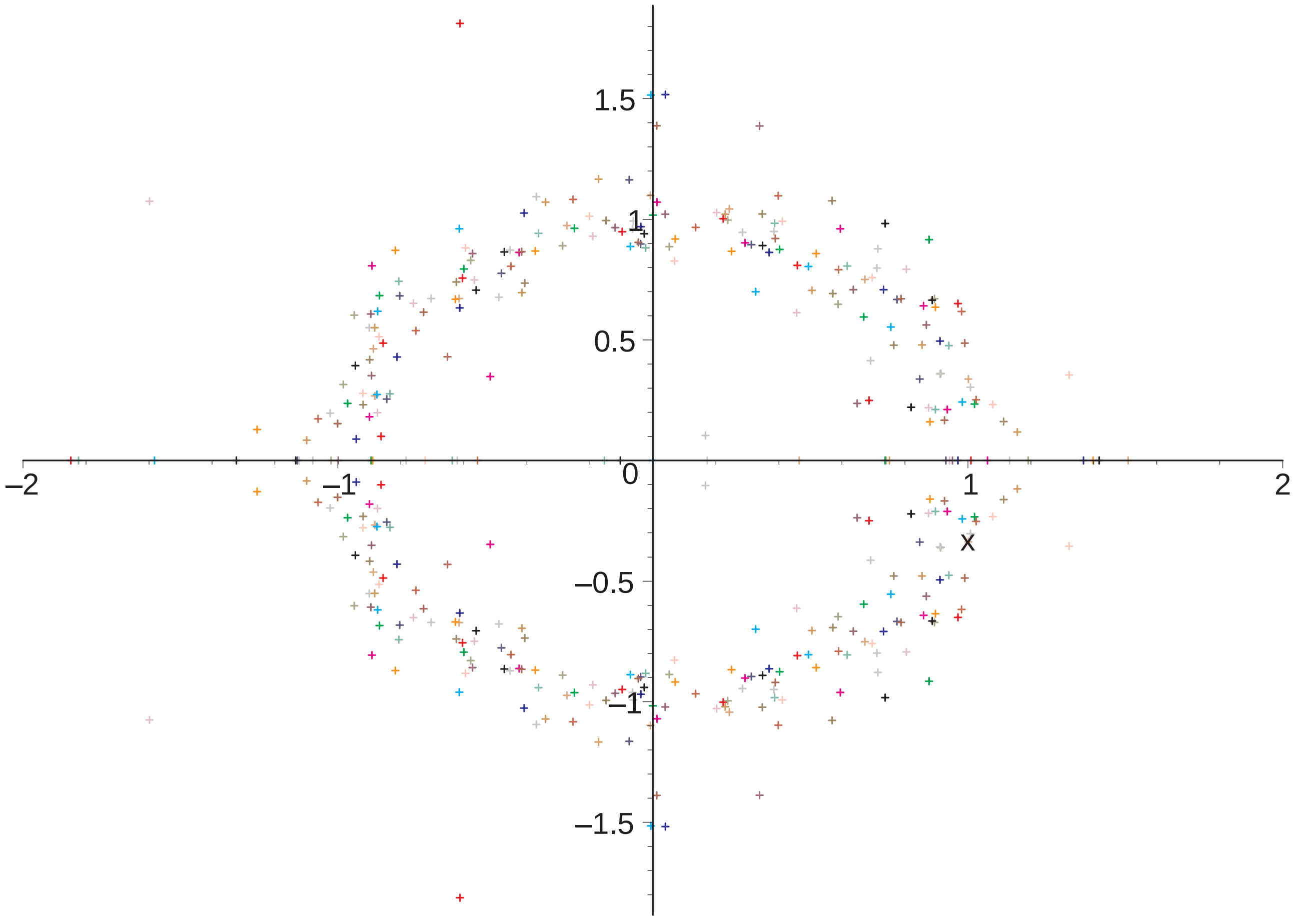
$$\Updownarrow$$

Sylvester-structure preserving deformation to singularity

Root distribution 20 polynomials of degree 20

Uniformly distributed coefficients

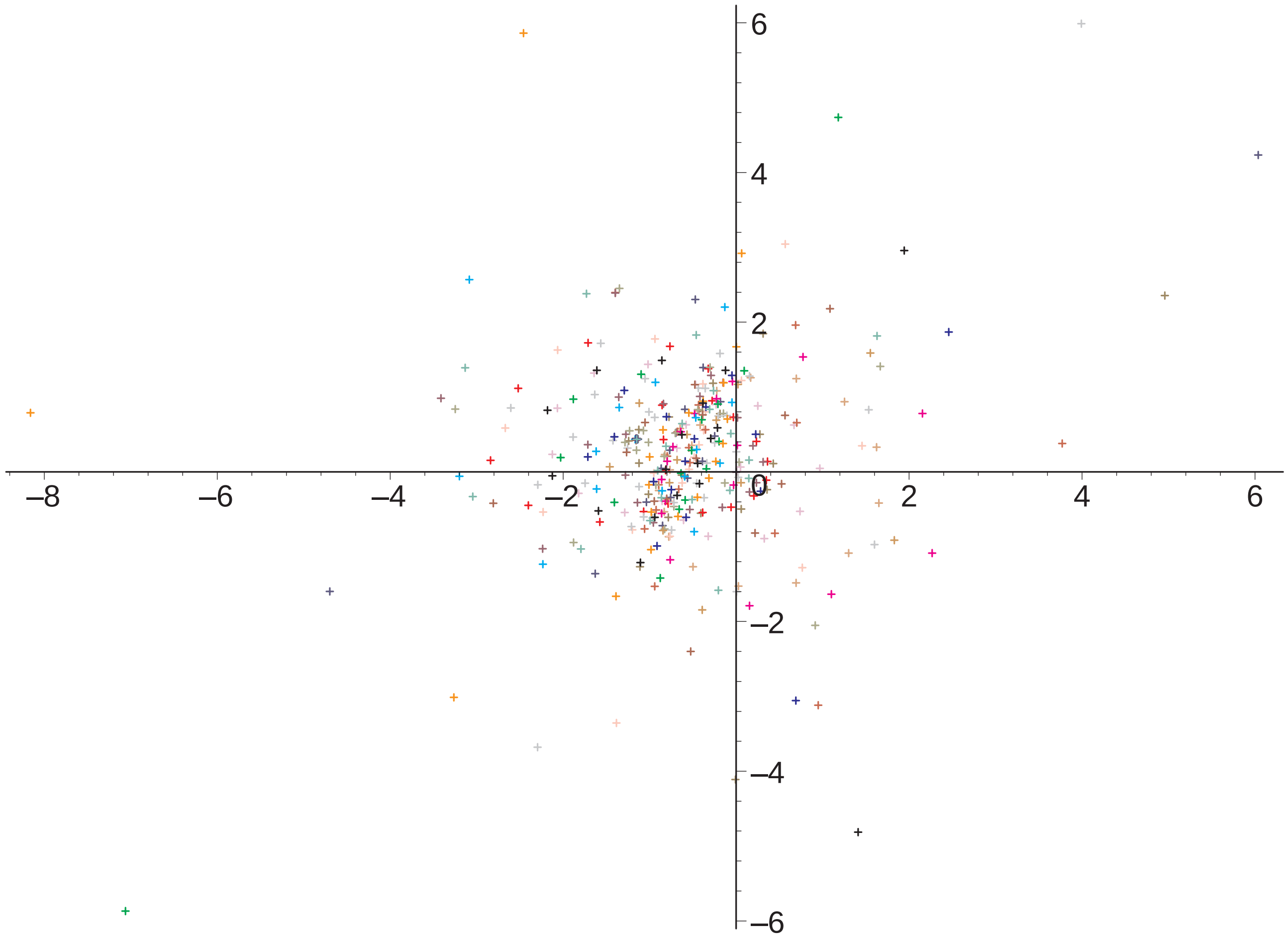
$\Longrightarrow$ small structured condition number



Root distribution 20 polynomials of degree 20

Binomially distributed coefficients

$\Longrightarrow$ large structured condition number



Conditioning of random projections

[Kaltofen and Trager 1988; Kaltofen and Yang 2007]

$$\frac{\sum_{j=1}^{t_f} a_j x_1^{d_{j,1}} (\xi_2 x_1 + \eta_2)^{d_{j,2}} \cdots (\xi_n x_1 + \eta_n)^{d_{j,n}}}{\sum_{k=1}^{t_g} b_k x_1^{e_{k,1}} (\xi_2 x_1 + \eta_2)^{e_{k,2}} \cdots (\xi_n x_1 + \eta_n)^{e_{k,n}}}, \quad \xi_i, \eta_i \in S \text{ random}$$

large structured condition numbers of Sylvester matrices

[Zippel 1979; ours here]

$$\frac{\sum_{j=1}^{t_f} a_j x_1^{d_{j,1}} \xi_2^{d_{j,2}} \cdots \xi_n^{d_{j,n}}}{\sum_{k=1}^{t_g} b_k x_1^{e_{k,1}} \xi_2^{e_{k,2}} \cdots \xi_n^{e_{k,n}}}, \quad \xi_i \in S \text{ random}$$

small structured condition numbers of Sylvester matrices

Well-conditioning of arising Fourier matrices to be established.

Table of Numerical Interpolation

Ex.	*Random Noise*	d_f, d_g	t_f, t_g	n	N	*error (ZNIPR)*
1	$10^{-5} \sim 10^{-3}$	1, 1	2, 2	2	136	9.46659e–7
2	$10^{-5} \sim 10^{-3}$	2, 2	3, 3	2	140	9.98831e–7
3	$10^{-5} \sim 10^{-3}$	1, 4	2, 4	3	233	3.70021e–7
4	$10^{-6} \sim 10^{-4}$	5, 2	10, 6	3	308	2.38743e–8
5	$10^{-7} \sim 10^{-5}$	7, 7	25, 25	5	1096	1.66383e–8
6	$10^{-7} \sim 10^{-5}$	10, 3	15, 5	8	2553	2.5896 e–5
8	$10^{-7} \sim 10^{-5}$	20, 20	7, 7	15	3107	1.90755e–9
9	$10^{-8} \sim 10^{-6}$	30, 30	6, 6	20	5213	3.77521e–12
11	$10^{-8} \sim 10^{-6}$	60, 60	7, 7	4	1987	5.31254e–12
12	$10^{-8} \sim 10^{-6}$	80, 80	6, 6	10	5127	5.10492e–11
13	$10^{-8} \sim 10^{-6}$	60, 0	6, 1	20	2862	2.00141e–12

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Fast matrix multiplication

Strassen's [1969] $O(n^{2.81})$ matrix multiplication algorithm

$$m_1 \leftarrow (a_{1,2} - a_{2,2})(b_{2,1} - b_{2,2})$$
$$m_2 \leftarrow (a_{1,1} + a_{2,2})(b_{1,1} + b_{2,2})$$
$$m_3 \leftarrow (a_{1,1} - a_{2,1})(b_{1,1} + b_{1,2})$$
$$m_4 \leftarrow (a_{1,1} + a_{1,2})b_{2,2} \quad \Big| \quad a_{1,1}b_{1,1} + a_{1,2}b_{2,1} = m_1 + m_2 - m_4 + m_6$$
$$m_5 \leftarrow a_{1,1}(b_{1,2} - b_{2,2}) \quad \Big| \quad a_{1,1}b_{1,2} + a_{1,2}b_{2,2} = m_4 + m_5$$
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$$m_7 \leftarrow (a_{2,1} + a_{2,2})b_{1,1} \quad \Big| \quad a_{2,1}b_{1,2} + a_{2,2}b_{2,2} = m_2 - m_3 + m_5 - m_7$$

Coppersmith and Winograd [1990]: $O(n^{2.38})$

Problems reducible to matrix multiplication:
linear system solving, determinant [Bunch and Hopcroft 1974],
characteristic polyn. [Keller-Gehrig'85, Pernet&Storjohann'07],
rational canonical form [Giesbrecht'92],
factoring in $\mathbb{Z}_2[x]$ [Berlekamp'69, Kaltofen and Shoup'95]

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rational canonical form [Giesbrecht'92],
factoring in $\mathbb{Z}_2[x]$: $n^{1.5+o(1)}$ [Umans'07, w/o FMM]

Matrix preconditioners

Theorem [Kaltofen and Saunders '91]

Let $A \in \mathsf{K}^{n\times n}$ with $r = \mathrm{rank}(A)$, $S \subseteq K$ with $|S| < \infty$.

$$\text{Probab.}\Bigl(\text{the first } r \text{ rows of } \begin{bmatrix} 1 & u_2 & u_3 & \dots & u_n \\ 0 & 1 & u_2 & \dots & u_{n-1} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & u_2 \\ 0 & \dots & & 0 & 1 \end{bmatrix} \cdot A$$

$$\text{are lin. indep.} \mid u_i \in S \text{ uniformly randomly}\Bigr)$$

$$\geq 1 - \frac{r}{|S|}$$

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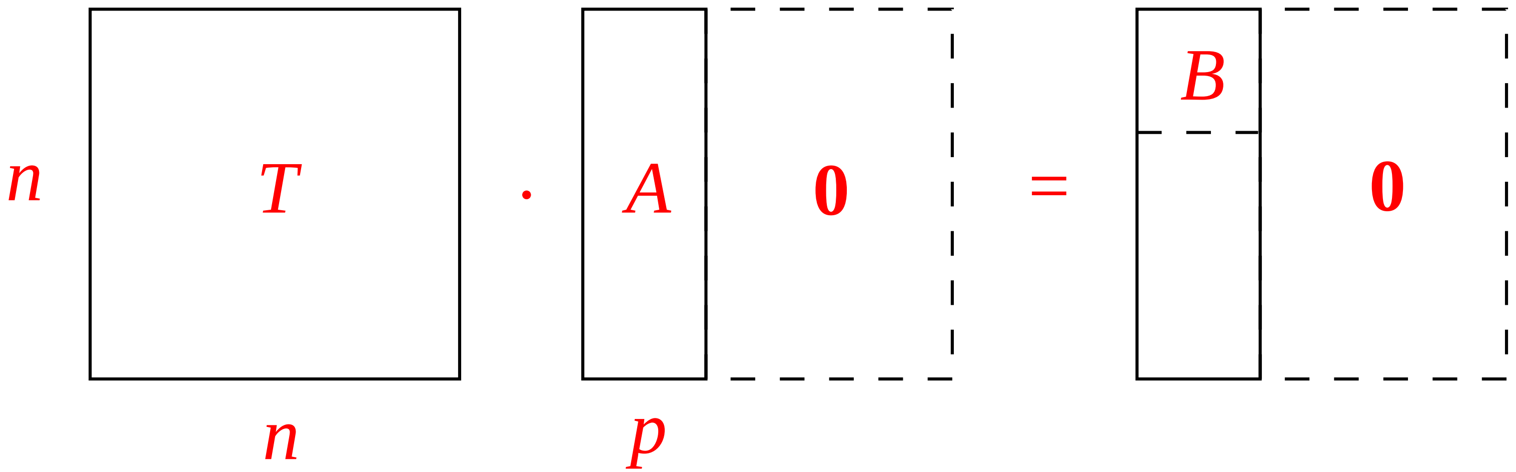
Note: Toeplitz times vector costs $O(n \log n)$ ops. (w. roots of unity)

Application: solving an overdetermined system $Ax = b$



With high probability, B has rank of A, so solve $Bx = c = \begin{bmatrix} (Tb)_1 \\ \vdots \\ (Tb)_p \end{bmatrix}$

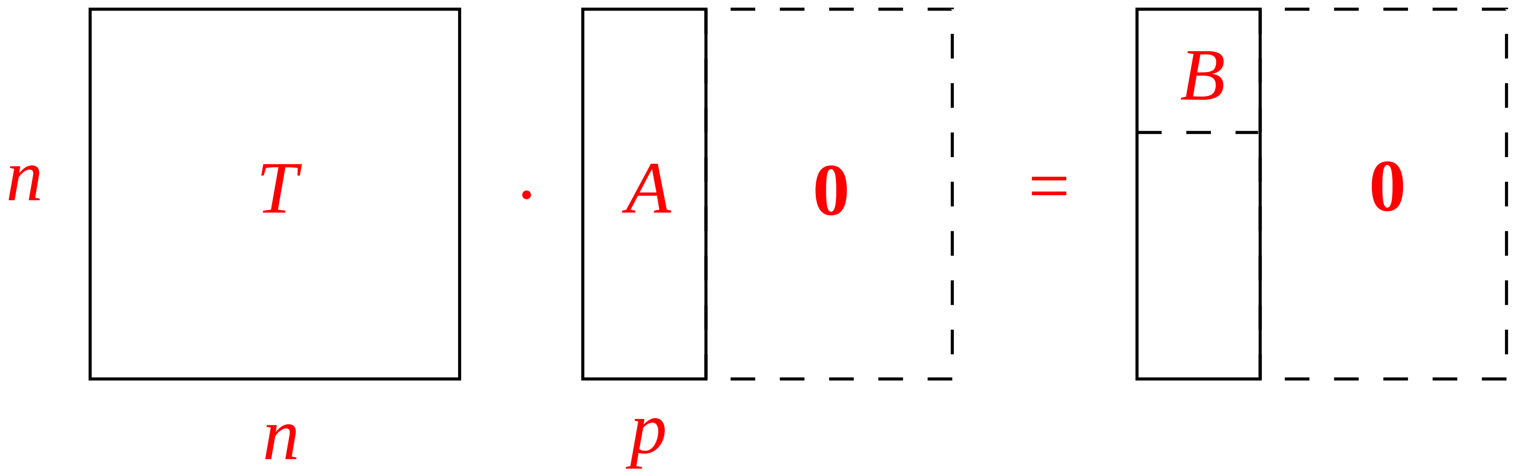
Arithmetic cost: $TA, Tb:$ $O(pn\log(n))$

solve $Bx = c:$ $O(p^3)$ or $O(p^{2.38})$

check $Ax = b:$ $O(pn)$

For $p = O(\sqrt{n\log(n)})$ or $O((n\log(n))^{0.72})$ the cost is $O(pn\log(n))$

Application: solving an overdetermined system $Ax = b$



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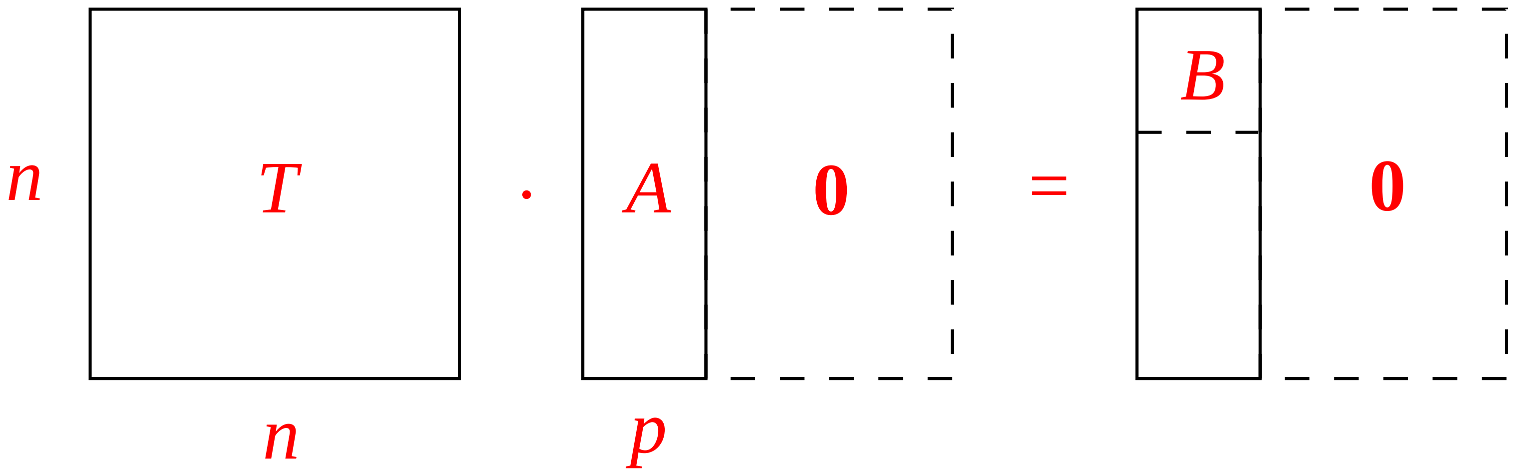
(Previously: $O(p^2 n)$ or $O(p^{1.38} n)$)

Application: solving an overdetermined system $Ax = b$



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For $p = O(\sqrt{n\log(n)})$ or $O((n\log(n))^{0.72})$ the cost is $O(pn\log(n))$
Highly over/underdet. systems can be solved **essentially optimally.**

Cond. Number Distr. of Precond.

Bryc, Włodzimierz, Dembo, Amir, and Jiang, Tiefeng. Spectral measure of large random Hankel, Markov and Toeplitz matrices. *Ann. Probab.*, 34(1):1–38, 2006.

Kaltofen, Zhi $\longrightarrow$ Bryc $\longrightarrow$ Zhidong Bai $\longrightarrow$ Bingyu Li,
Jack Silverstein

The distributions are tricky (we so far have no separation from zero theorem)!

Danke schön!

Thank you!