Hybrid Symbolic-Numeric Computation

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What is Symbolic-Numeric Computation?
Definition by Example

Corless et al. ISSAC'95–Karmarkar and Lakshman '96
Nearest approximate GCD in the l^2-norm:

Let $f, g \in \mathbb{C}[z]$, both monic, $\deg(f) = m$ and $\deg(g) = n$.
Assuming that $\gcd(f, g) = 1$, find $\tilde{f}, \tilde{g} \in \mathbb{C}[z]$, s.t.
$\gcd(\tilde{f}, \tilde{g})$ is non-trvial and
$\mathcal{N} = \|f - \tilde{f}\|^2 + \|g - \tilde{g}\|^2$ is minimized.

$\|f\|$ denotes a norm of the coefficient vector of f.

The *symbolic* minimum of $\mathcal{N}$ with respect to a common root $\alpha \in \mathbb{C}$ can be obtained in closed-form:

$$\mathcal{N}_{min} = \frac{\overline{f(\alpha)} f(\alpha)}{\sum_{k=0}^{m-1} (\overline{\alpha}\alpha)^k} + \frac{\overline{g(\alpha)} g(\alpha)}{\sum_{k=0}^{n-1} (\overline{\alpha}\alpha)^k}$$

The individual perturbations of the coefficients of f and g are

$$f_i - \tilde{f}_i = \frac{(\overline{\alpha})^i f(\alpha)}{\sum_{k=0}^{m-1} (\overline{\alpha}\alpha)^k} \quad \text{and} \quad g_j - \tilde{g}_j = \frac{(\overline{\alpha})^j g(\alpha)}{\sum_{k=0}^{n-1} (\overline{\alpha}\alpha)^k}$$

for $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$, respectively.

Corless et al. 1995: use numerical optimizer on $\mathcal{N}(u + \boldsymbol{i} v)$

Lakshman and Karmarkar 1996: use symbolic optimizer on $\mathcal{N}(u + \boldsymbol{i} v)$; give expressions for k common root

Show Maple worksheets

Derivation of formulas

The **Reduced Problem:** Given $f \in \mathbb{C}[z]$ and $\alpha \in \mathbb{C}$. Find $\tilde{f} \in \mathbb{C}[z]$, s.t.

$$\tilde{f}(\alpha) = 0, \quad \text{and} \quad \|f - \tilde{f}\| = \min .$$

Let

$$\begin{aligned} f(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \\ \tilde{f}(z) &= (z-\alpha) \sum_{k=0}^{n-1} u_k z^k \\ &= u_{n-1} z^n + (u_{n-2} - \alpha) z^{n-1} + (u_{n-3} - \alpha u_{n-2}) z^{n-2} + \\ &\quad \cdots \quad + (u_0 - \alpha u_1) z - \alpha u_0 \end{aligned}$$

In terms of linear algebra:

$$\| f - \tilde{f} \| = \min_{\mathbf{u} \in \mathbb{C}^n} \| \mathbf{Pu} - \mathbf{b} \| \tag{1}$$

$$\begin{aligned} \mathbf{b} &= [a_0, \ldots, a_{n-1}, a_n]^{tr} \in \mathbb{C}^{n+1} \\ \mathbf{u} &= [u_0, \ldots, u_{n-1}]^{tr} \in \mathbb{C}^n \end{aligned}$$

$$\mathbf{P} = \begin{bmatrix} -\alpha & & & \mathbf{0} \\ 1 & -\alpha & & \\ & \ddots & \ddots & \\ \mathbf{0} & & 1 & -\alpha \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{(n+1)\times n} \tag{2}$$

$$\mathbf{v} = [1\ \alpha\ \alpha^2\ \ldots] \perp \text{col-space}(\mathbf{P}); \text{ortho.proj.} = \frac{(\mathbf{v}, \mathbf{b})}{(\mathbf{v}, \mathbf{v})} \mathbf{v} \tag{3}$$

(1) is an over-determined linear system of equations.

LP problem, if $\|\cdot\|$ is the $\begin{cases} l^\infty & \text{norm, or} \\ l^1 & \text{norm} \end{cases}$

LS problem, if $\|\cdot\|$ is the l^2 norm.

Solutions for the l^2-norm in closed form:

$$\mathcal{N}_{min}(\alpha) = \|f - \tilde{f}\|^2 = \frac{\overline{f(\alpha)}\, f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k}$$

$$f_j - \tilde{f}_j = \frac{(\overline{\alpha})^j f(\alpha)}{\sum_{k=0}^{n} (\overline{\alpha}\alpha)^k},\ 0 \le j \le n$$

Hybrid Symbolic-Numeric Algorithms

Hybridization of Sylvester structure and structure preserving total least squares algorithms

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Other: Approx. factorization: Ruppert/Gao + SVD/STLS
Approx. sparse interpolation: Ben-Or/Tiwari + Prony
Approx. polynomial relations: Buchberger/Möller + SVD
Approx. polynomial system solving: Auzinger/Stetter
multiplication matrices $[g] \mapsto [fg]$ + eigenvectors

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- Journal of Symbolic Computation