Approximate Greatest Common Divisors of Several Multivariate Polynomials with Linearly Constrained Coefficients and Singular Polynomials

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Approximate GCD Problem

Given polynomials $f_1, \ldots, f_s \in F[y_1, y_2, \ldots, y_r] \setminus \{0\}$, where F is $\mathbb{R}$ or $\mathbb{C}$; let $d_i = \mathrm{tdeg}(f_i)$ and $k \leq d_i$ for all i with $1 \leq i \leq s$.
We wish to compute $\Delta f_1, \ldots, \Delta f_s \in F'[y_1, y_2, \ldots, y_r]$, where F' is $\mathbb{R}$ or $\mathbb{C}$, such that $\mathrm{tdeg}(\Delta f_1) \leq d_1, \ldots, \mathrm{tdeg}(\Delta f_s) \leq d_s$,

- $\mathrm{tdeg}(\mathrm{gcd}(f_1 + \Delta f_1, \ldots, f_s + \Delta f_s)) \geq k$,
- $\|\Delta f_1\|_2^2 + \cdots + \|\Delta f_s\|_2^2$ is minimized.

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Problem depends on choice of norm $\|\cdot\|$, and notion of degree.
We use 2-norm on the coefficient vector, and total degree.

Previous Work on Approximate GCD

- Modified Euclidean algorithm for polynomials with floating point coefficients
  [Dunaway ’74, Schönhage ’85, Sasaki and Noda ’89 & ’91, Ochi et al. ’91, Hribernig and Stetter ’97, Beckermann and Labahn ’98, Sasaki and Sasaki ’01, Sanuki ’05]
- Nearby roots matching, resultant-based algorithms, QR factorization, Hensel lifting strategy [Pan ’01, Emiris et al. ’96, Rupprecht ’99, Zhi and Noda ’00, Corless et al. ’04]
- Least squares and SVD-based total least squares methods
  [Corless et al. ’95, Chin et al. ’98, Karmarkar and Lakshamn ’96 & ’98, Zeng ’03 & ’04, Gao et al. ’04, Diaz-Toca and Gonzalez-Vega ’02 & ’06]
- Structure preserving total least squares algorithms
  [Chu et al. ’03, Li et al. ’05, Kaltofen et al. ’05, Botting et al. ’05, Markovsky and Huffel ’05]

Globally Nearest Complex/Real Solution

Theorem 1. Let $f_1,\dots,f_s \in F[y_1,\dots,y_r]\setminus\{0\}$, where F is $\mathbb{C}$ or $\mathbb{R}$, $d_i = \mathrm{tdeg}(f_i)$ and $k \le d_i$ with $1 \le i \le s$. There exist $\widehat{f}_i \in F[y_1,y_2,\dots,y_r], 1 \le i \le s$ with

$$\mathrm{tdeg}(\widehat{f}_i) \le d_i, \;\; \text{and } \mathrm{tdeg}(\mathrm{gcd}(\widehat{f}_1,\dots,\widehat{f}_s)) \ge k,$$

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such that for all $\overline{f}_i \in F[y_1, y_2, \ldots, y_r], 1 \le i \le s$ with

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we have

$$\|\widehat{f}_1 - f_1\|_2^2 + \cdots + \|\widehat{f}_s - f_s\|_2^2 \le \|\overline{f} - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2.$$

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we have

$$\|\widehat{f_1}-f_1\|_2^2+\cdots+\|\widehat{f_s}-f_s\|_2^2 \le \|\overline{f}-f_1\|_2^2+\cdots+\|\overline{f}_s-f_s\|_2^2.$$

Remark: Theorem is false if $\mathrm{tdeg}\,\gcd = k$.

Multi-polynomial Generalized Sylvester Matrix

Denote the coefficient matrix of the system

$$u_1 \neq 0, \quad \forall i, 2 \leq i \leq s: u_i f_1 + u_1 f_i = 0, \quad \mathrm{tdeg}(u_i) \leq d_i - k$$

as $S_k(f_1, \ldots, f_s) =$

$$\begin{bmatrix} \mathrm{C}^{[d_2-k]}(f_1) & \mathbf{0} & \ldots & \mathbf{0} & \mathrm{C}^{[d_1-k]}(f_2) \\ \mathbf{0} & \mathrm{C}^{[d_3-k]}(f_1) & & \mathbf{0} & \mathrm{C}^{[d_1-k]}(f_3) \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathrm{C}^{[d_s-k]}(f_1) & \mathrm{C}^{[d_1-k]}(f_s) \end{bmatrix}$$

Convolution Matrix

The convolution matrix $\mathrm{C}^{[l]}(f)$ produces the coefficient vector of $u \cdot f$ as $\mathrm{C}^{[l]}(f) \cdot \vec{u}$, where $l = \mathrm{tdeg}(u)$. For instance,

$$\mathrm{C}^{[2]}(a_2 y^2 + a_1 y + a_0) \cdot \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2 & 0 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{bmatrix} \cdot \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}.$$

In the univariate case, the matrix is of Toeplitz form. In the multivariate case, the dimensions of $\mathrm{C}^{[l]}(f)$ with $\mathrm{tdeg}(f) = m$ are $\binom{l+m+r}{r} \times \binom{l+r}{r}$.

The Minimal Perturbation Problem

Lemma 1. $\mathrm{tdeg}(\gcd(f_1,\ldots,f_s)) \geq k$ iff $S_k(f_1,\ldots,f_s)$ has rank deficiency at least one.

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Our problem can be transformed into:

$$\min_{\operatorname{tdeg}(\gcd(\overline{f}_1,\ldots,\overline{f}_s))\geq k} \|\overline{f}_1 - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2$$

$$\iff \min_{\dim \operatorname{Nullspace}(\overline{S}_k)\geq 1} \|\overline{f}_1 - f_1\|_2^2 + \cdots + \|\overline{f}_s - f_s\|_2^2$$

where $\overline{S}_k$ is the k-th Sylvester matrix generated by $\overline{f}_1,\ldots,\overline{f}_s$ with $\operatorname{tdeg}\overline{f}_i \leq d_i, 1 \leq i \leq s.$

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where $\overline{S}_k$ is the k-th Sylvester matrix generated by $\overline{f}_1,\ldots,\overline{f}_s$ with $\mathrm{tdeg}\,\overline{f}_i \leq d_i, 1 \leq i \leq s$.

Nearest singular matrix + Sylvester structure

$\Longrightarrow$ Approximate GCD

Structure Preserving Low Rank Approximation

Let $S(\zeta) = [A_1(\zeta) \mid b(\zeta) \mid A_2(\zeta)]$ and let $A(\zeta) = [A_1(\zeta) \mid A_2(\zeta)]$, where ζ contains the coefficients of $f_1, \ldots, f_s$. We solve the structure-preserving total least norm problem

$$\min_{\Delta\mathbf{c}\in\mathbb{R}^\nu} \|\Delta\mathbf{c}\| \text{ or } \min_{\Delta\mathbf{c}\in\mathbb{C}^\nu} \|\Delta\mathbf{c}\| \quad \text{with} \quad A(\mathbf{c}+\Delta\mathbf{c})\mathbf{x} = b(\mathbf{c}+\Delta\mathbf{c})$$

for some vector $\mathbf{x}$. Here $\mathbf{c}$ is fixed to the initial coefficient vector.

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Remark: We choose the column $b(\zeta)$ corresponding to the absolutely largest component in the first singular vector of $S(\zeta)$.

The STLN Algorithm

The STLN Algorithm first initializes $\mathbf{x}$ as a solution to

$$A(\mathbf{c})\mathbf{x} \approx b(\mathbf{c})$$

for the input parameters $\mathbf{c}$ and $\Delta\mathbf{c} = \mathbf{z} = 0,$ then updates $\mathbf{x} + \Delta\mathbf{x}$ and $\mathbf{z} + \Delta\mathbf{z}$ satisfy

$$\min \|\mathbf{z} + \Delta\mathbf{z}\| \quad \text{with} \quad A(\mathbf{c} + \mathbf{z} + \Delta\mathbf{z})\,(\mathbf{x} + \Delta\mathbf{x}) \approx b(\mathbf{c} + \mathbf{z} + \Delta\mathbf{z}).$$

References: Rosen et al. ’96, Park et al. ’99

New Initialization of $\mathbf{x}$ and $\Delta\mathbf{c}$

Suppose $H(\xi)\zeta = S(\zeta)\xi$ and $\mathbf{v}$ is the first singular vector of $S(\mathbf{c})$ and

$$\Delta\mathbf{c} = \mathbf{z} = -H(\mathbf{v})^{Tr}(H(\mathbf{v})H(\mathbf{v})^{Tr})^{-1}S(\mathbf{c})\mathbf{v}.$$

We have $-S(\mathbf{z})\mathbf{v} = -H(\mathbf{v})\mathbf{z} = S(\mathbf{c})\mathbf{v}$, hence $S(\mathbf{c}+\mathbf{z})\mathbf{v} = 0$.

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We initialize $\mathbf{x}$ as

$$\mathbf{x} = \left[-\frac{\mathbf{v}[1]}{\mathbf{v}[t]}, \ldots, -\frac{\mathbf{v}[t-1]}{\mathbf{v}[t]}, -\frac{\mathbf{v}[t+1]}{\mathbf{v}[t]}, \ldots\right]^{Tr},$$

where $\mathbf{v}[t]$ is the absolutely largest component. We have

$$A(\mathbf{c}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\mathbf{z}).$$

Reference: Lemmerling'99

Penalty Method

Let the residue

$$\mathbf{r}(\mathbf{z}+\Delta\mathbf{z},\mathbf{x}+\Delta\mathbf{x}) = b(\mathbf{c}+\mathbf{z}+\Delta\mathbf{z}) - A(\mathbf{c}+\mathbf{z}+\Delta\mathbf{z})\,(\mathbf{x}+\Delta\mathbf{x})$$

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The structured minimization problem can be transformed into:

$$\min_{\Delta\mathbf{z},\Delta\mathbf{x}} \left\| \begin{matrix} w\mathbf{r}(\mathbf{z}+\Delta\mathbf{z},\mathbf{x}+\Delta\mathbf{x}) \\ \mathbf{z}+\Delta\mathbf{z} \end{matrix} \right\|,$$

where w is a large penalty value between 10^8 and 10^{10}.

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Linear Approximation to the Residue

$$\mathbf{r}(\mathbf{z}+\Delta\mathbf{z},\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{r}(\mathbf{z},\mathbf{x}) + b(\Delta\mathbf{z}) - A(\mathbf{c}+\mathbf{z})\Delta\mathbf{x} - A(\Delta\mathbf{z})\mathbf{x}.$$

We introduce a constant matrix P with

$$b(\Delta\mathbf{z}) = P\Delta\mathbf{z},$$

and a Sylvester-like matrix $Y(\xi)$ such that

$$Y(\xi)\zeta = A(\zeta)\xi, \quad \text{in particular} \quad A(\Delta\mathbf{z})\mathbf{x} = Y(\mathbf{x})\Delta\mathbf{z}.$$

Hence the first order approximation of the residue is

$$\mathbf{r}(\mathbf{z}+\Delta\mathbf{z},\mathbf{x}+\Delta\mathbf{x}) \approx \mathbf{r}(\mathbf{z},\mathbf{x}) + P\Delta\mathbf{z} - A(\mathbf{c}+\mathbf{z})\Delta\mathbf{x} - Y(\mathbf{x})\Delta\mathbf{z}.$$

First Order Iterative Update

$$\min_{\Delta\mathbf{x},\Delta\mathbf{z}} \left\| \begin{bmatrix} w(Y(\mathbf{x})-P) & wA(\mathbf{c}+\mathbf{z}) \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\mathbf{x} \end{bmatrix} + \begin{bmatrix} -w\mathbf{r}(\mathbf{z},\mathbf{x}) \\ \mathbf{z} \end{bmatrix} \right\|$$

The iterative update

$$\mathbf{x} = \mathbf{x} + \Delta\mathbf{x}, \quad \mathbf{z} = \mathbf{z} + \Delta\mathbf{z}$$

is stopped when

$$\|\Delta\mathbf{x}\| \leq tol \quad \text{and/or} \quad \|\Delta\mathbf{z}\| \leq tol$$

Random Complex Perturbation in the Initialization

We initialize $\mathbf{z}$ as:

$$\mathbf{z} = -H(\mathbf{v})^{Tr}(H(\mathbf{v})H(\mathbf{v})^{Tr})^{-1}S(\mathbf{c}+\boldsymbol{i}\Delta\mathbf{c}_{rand})\mathbf{v},$$

where $\mathbf{v}$ is the first sing. vec. of $S(\mathbf{c}+\boldsymbol{i}\Delta\mathbf{c}_{rand})$, $\Delta\mathbf{c}_{rand}$ is a random real vector of small noise.

We initialize $\mathbf{x}$ by normalizing $\mathbf{v}$, as before then

$$A(\mathbf{c}+\boldsymbol{i}\Delta\mathbf{c}_{rand}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\boldsymbol{i}\Delta\mathbf{c}_{rand}+\mathbf{z}).$$

Multiple Local Minima

Consider the polynomials

$$f = 1000y^{10} + y^3 - 1 \quad \text{and} \quad g = y^2 - \frac{1}{100}.$$

We seek to compute the nearest pair of polynomials $\tilde{f}$ and $\tilde{g}$ that have a non-trivial GCD.

The algorithm converges after about ten iterations in average to the local minima:

$$0.0421579, 0.0463113, 0.0474087, 0.0493292, \ldots$$

for different initializations.

Among solutions, the polynomials

$$\tilde{f} = 1000.0y^{10} + 0.0000147908y^9 + \cdots - 0.991601,$$
$$\tilde{g} = 0.956139y^2 - 0.0887590y - 0.189618,$$

have a common divisor

$$y - 0.4941547,$$

and the backward error is

$$\|f - \tilde{f}\|_2^2 + \|g - \tilde{g}\|_2^2 = 0.0421579.$$

It is the non-monic global minimum found by the global methods

Karmarkar and Lakshman Y.N. 1998; Hitz and Kaltofen 1998.

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Sylvester $\neq$ Toeplitz à la Rump [2003]

App. GCD of Multivariate Polynomials

Ex.	d_i	k	e	*it.*	*error (Zeng)*	*error (GKMYZ)*	*error (STLN)*
1	7,7	4	3	2	2.44360e–4	2.59476e–4	6.50358e–5
2	7,7	4	5	1	2.44404e–8	2.59194e–8	6.50357e–9
3	7,7	4	7	1	2.44405e–12	2.59191e–12	6.50357e–13
4	7,7	4	9	1	2.44396e–16	2.59187e–16	6.50361e–17
5	6,6	3	2	3	2.26617	1.49524	4.80154e–1
6	10,10	5	4	2		2.74672e–3	1.84914e–3
7	8,8	4	5	2	7.09371e–5	2.38059e–5	2.01393e–5
8	40,40	30	5	2	1.39858e–3	4.83931e–4	4.39489e–4
9	10,9,8	5	3	2			6.21772e–2
10	8,7,8,6	4	5	2			4.04458e–6

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  where λ is an integer.

- In general, the linear constraints on the goal coefficients are

$$\Gamma\zeta = \gamma.$$

Intuitive Approach

We add the constraints $\Gamma(\mathbf{c}+\mathbf{z}) = \gamma$ directly, and iterate on

$$\min_{\Delta\mathbf{x},\Delta\mathbf{z}} \left\| \begin{bmatrix} w(Y(\mathbf{x})-P) & wA(\mathbf{c}+\mathbf{z}) \\ \mathbf{I} & \mathbf{0} \\ w\Gamma & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z} \\ \Delta\mathbf{x} \end{bmatrix} + \begin{bmatrix} -w\mathbf{r}(\mathbf{z},\mathbf{x}) \\ \mathbf{z} \\ w(\Gamma(\mathbf{c}+\mathbf{z})-\gamma) \end{bmatrix} \right\|.$$

However, that least squares problem has a larger dimension.

Our Approach

Let the linear constraints on the goal coefficients be

$$\Gamma\zeta = \gamma.$$

We construct for the linear system (**c** is the input coeff. vec.)

$$\Gamma\zeta = \gamma - \Gamma\mathbf{c}$$

a matrix C, a vector **d** and a sub-vector of free parameters $[\zeta_{i_1}, \ldots, \zeta_{i_\mu}]^{Tr}$ such that

$$\zeta = C\zeta^{-} + \mathbf{d}, \quad \zeta^{-} = \begin{bmatrix} \zeta_{i_1} \\ \vdots \\ \zeta_{i_\mu} \end{bmatrix}; \quad \Gamma C = 0, \quad \Gamma\mathbf{d} = \gamma - \Gamma\mathbf{c}.$$

Initialization of $\mathbf{x}$ and $\mathbf{z}$

Let

$$\mathbf{z} = \mathbf{d} - C\,(H(\mathbf{v})C)^{Tr}(H(\mathbf{v})C\,(H(\mathbf{v})C)^{Tr})^{-1}S(\mathbf{c}+\mathbf{d})\mathbf{v},$$

$\mathbf{v}$ is the first singular vector of the matrix $S(\mathbf{c}+\mathbf{d})$. We have

$$S(\mathbf{c}+\mathbf{z})\mathbf{v} = 0.$$

We initialize $\mathbf{x}$ as

$$\mathbf{x} = \left[-\frac{\mathbf{v}[1]}{\mathbf{v}[t]}, \ldots, -\frac{\mathbf{v}[t-1]}{\mathbf{v}[t]}, -\frac{\mathbf{v}[t+1]}{\mathbf{v}[t]}, \ldots\right]^{Tr},$$

where $\mathbf{v}[t]$ is the absolutely largest component. We have

$$A(\mathbf{c}+\mathbf{z})\mathbf{x} = b(\mathbf{c}+\mathbf{z}).$$

First Order Iterative Update

The iterative update can be written as

$$\min_{\Delta \mathbf{x}, \Delta \mathbf{z}^-} \left\| \begin{bmatrix} w(Y(\mathbf{x}) - P)C & wA(\mathbf{c} + \mathbf{z}) \\ C & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta \mathbf{z}^- \\ \Delta \mathbf{x} \end{bmatrix} + \begin{bmatrix} -w\mathbf{r}(\mathbf{z}, \mathbf{x}) \\ \mathbf{z} \end{bmatrix} \right\|.$$

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The new coefficient values are

$$\mathbf{c}+\mathbf{z}+\Delta\mathbf{z} = \mathbf{c}+\mathbf{z}+C\Delta\mathbf{z}^-,$$

and satisfies

$$\Gamma(\mathbf{c}+\mathbf{z}+\Delta\mathbf{z}) = \Gamma(\mathbf{c}+\mathbf{z}) = \Gamma(\mathbf{c}+\mathbf{d}) = \gamma$$

throughout the iteration.

Nearest Singular Polynomials

Lemma 3. Let $f(y) \in F[y]$ with $\deg(f) = n$ over a field F of characteristic 0, and let k be a multiplicity with $2 \leq k \leq n$. Denote by $f^{[i]} = \mathrm{d}^i f/\mathrm{d}y^i$, then $\deg(\gcd(f^{[0]}, \ldots, f^{[k-1]})) \geq 1$ iff the matrix $S_k^{sing}(f) =$

$$\begin{bmatrix} \mathrm{C}^{[n-1]}(f^{[k-1]}) & \mathbf{0} & \ldots & \mathbf{0} & \mathrm{C}^{[n-k]}(f^{[0]}) \\ \mathbf{0} & \mathrm{C}^{[n-2]}(f^{[k-1]}) & & \mathbf{0} & \mathrm{C}^{[n-k]}(f^{[1]}) \\ \vdots & & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathrm{C}^{[n-k+1]}(f^{[k-1]}) & \mathrm{C}^{[n-k]}(f^{[k-2]}) \end{bmatrix}$$

has rank deficiency at least one.

Weighted Minimization Problem

For the nearest k-fold singular polynomial one optimizes

$$\|\Delta f\|,$$

while the GCD problem with the corresponding constraints on the coefficients optimizes

$$\sum \|\mathrm{d}^i \Delta f / \mathrm{d}y^i\|,$$

which has a different minimum.

We introduce a weight matrix D to the minimization problem,

$$\min_{\Delta\mathbf{c}\in\mathbb{R}^\nu} \|D\Delta\mathbf{c}\| \text{ or } \min_{\Delta\mathbf{c}\in\mathbb{C}^\nu} \|D\Delta\mathbf{c}\| \quad \text{with} \quad A(\mathbf{c}+\Delta\mathbf{c})\mathbf{x} = b(\mathbf{c}+\Delta\mathbf{c}).$$

Then the first order iterative update becomes

$$\min_{\Delta\mathbf{x},\,\Delta\mathbf{z}^-} \left\| \begin{bmatrix} w(Y(\mathbf{x})-P)C & wA(\mathbf{c}+\mathbf{z}) \\ DC & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{z}^- \\ \Delta\mathbf{x} \end{bmatrix} + \begin{bmatrix} -w\mathbf{r}(\mathbf{z},\mathbf{x}) \\ D\mathbf{z} \end{bmatrix} \right\|.$$

Reference: H. Park et al. 99

Ex.	*d*	*k*	*it.*	*error(ZNKW)*	*error(STLN)*
1	4	2	12	.1763296120	.1763296118
		3	34	.6261127476	.6261127498
2	4	2	4	.1552760123e–12	.1552723415e–12
		3	11	.8834609009e–9	.9814886696e–9
		4	4	.2021848972e–4	.1958553174e–4
3	4	2	4	.1645037985e–10	.16450617515e–10
		3	4	.4144531274e–6	.4144531274e–6
		4	12	.1049993144	.1049993152
4	5	2	1	.2460987981e–8	.2461467456e–8
	5	3	20	.3681785214	.3681784856
5	6	2	2	.3231668276e–5	.3231668277e–5
6	6	2	3	.3009788845e–11	.3009789157e–11
		3	3	.7453849284e–6	.7453849284e–6
		4	24	.4449023547	.4449023547
7	5	2	8	.8565349347	.8565327605
8	21	2	2	.190477e–8	.1893347157e–8
		3	6	.963776e–4	.9637591989e–4

Univariate Singular Polynomials

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- Investigation of displacement operators for generalized Sylvester matrix or Bezout matrix or Ruppert matrix.
- How to certify a local minimum is a global minimum

Code + Benchmarks at:

`http://mmrc.iss.ac.cn/~lzhi/Research/hybrid/manystln.html`

or

`google->kaltofen` (click on “Software”)

Grazie mille!