Tellegen's principle and the synthesis of algorithms

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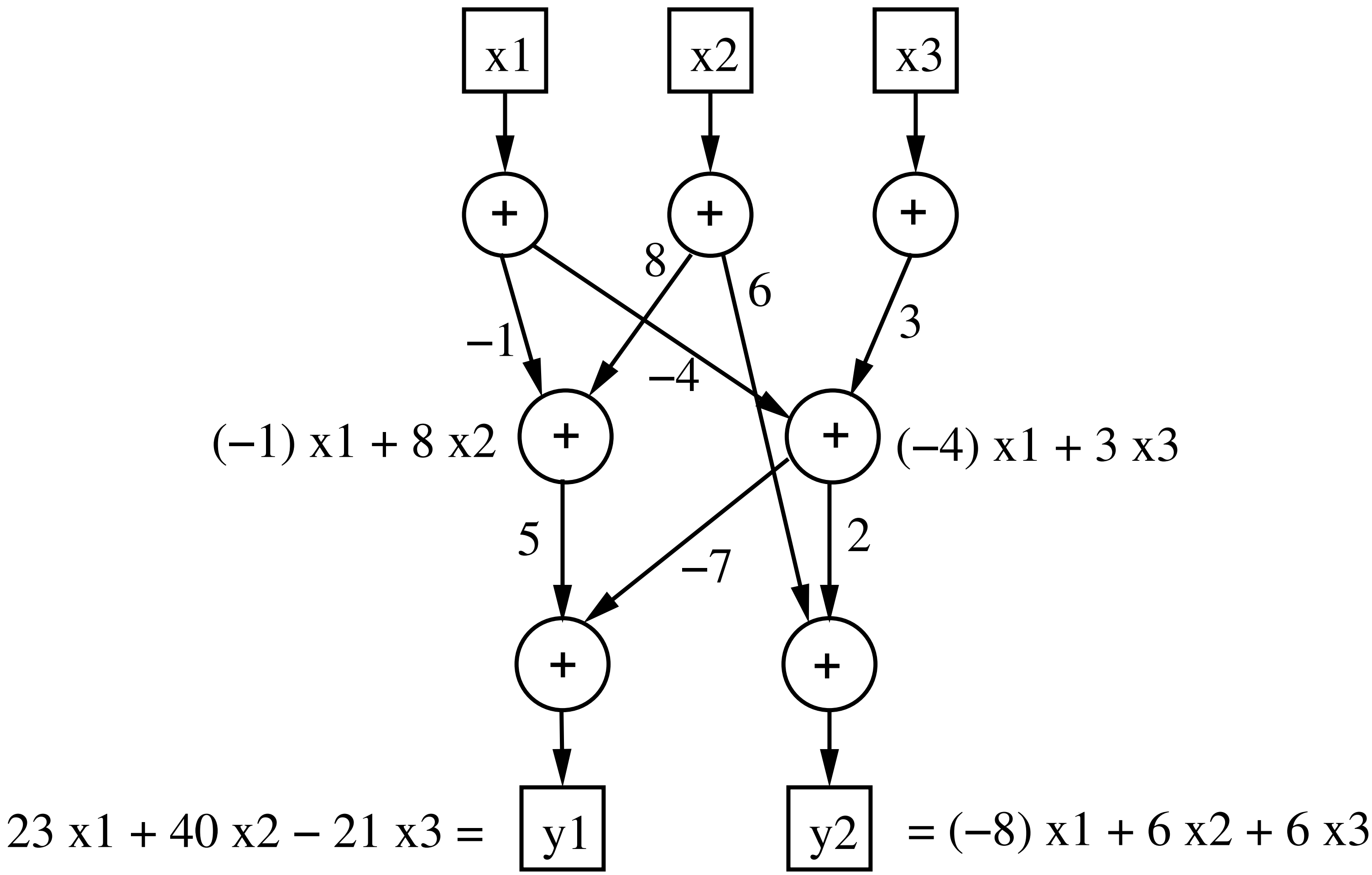
Tellegen's principle (Bordewijk's theorem)



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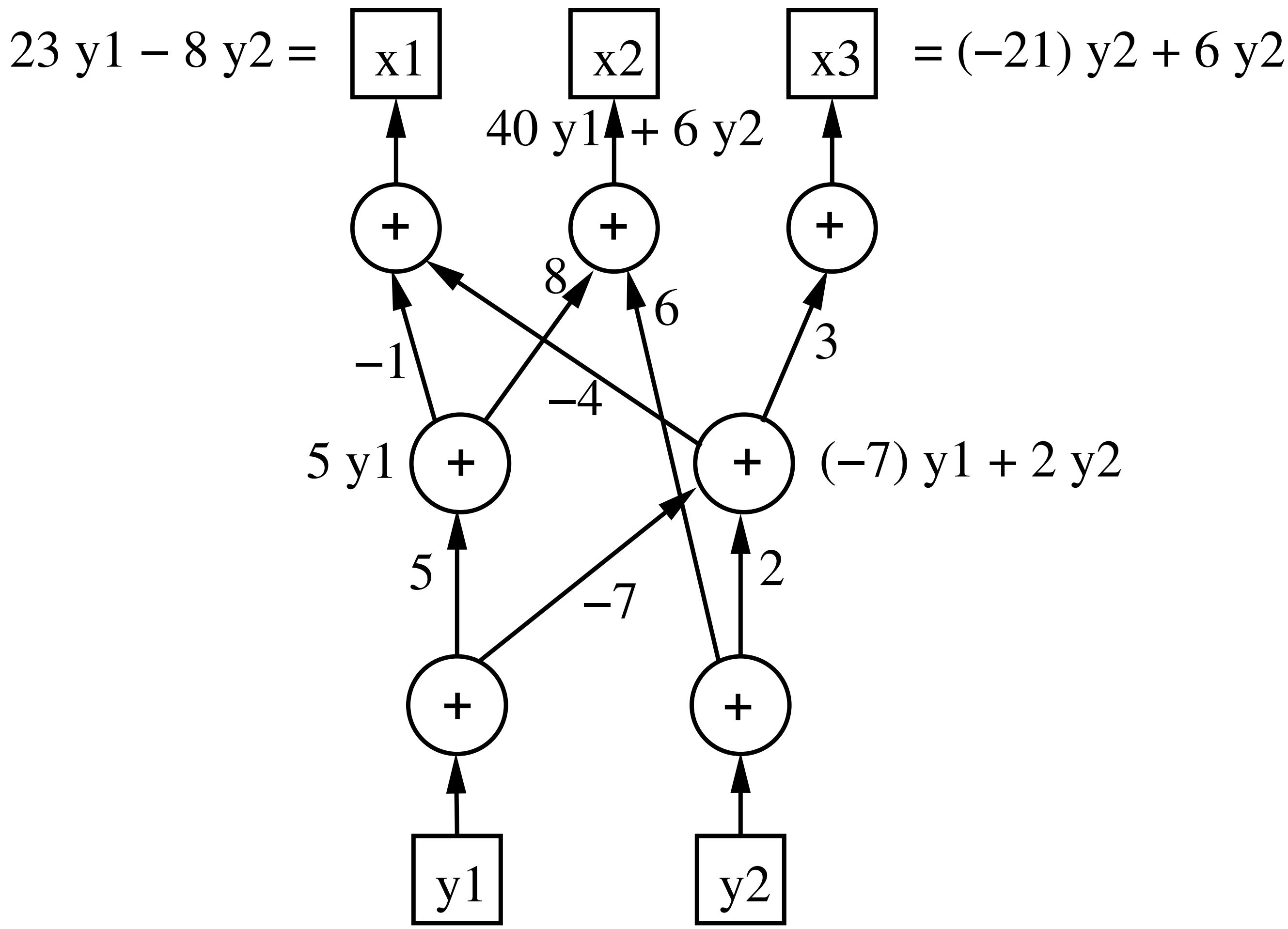

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A first application: weighted power sums

Input: $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ (weights)
Output: $b_i = x_1^i y_1 + \cdots + x_n^i y_n$ for all $i = 0, 1, \ldots, n-1$

$$\begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \ldots & x_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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= transposed multipoint polynomial evaluation
(transposed Vandermonde-matrix times a vector)

$$\begin{bmatrix} c_0 & c_1 & \ldots & c_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & \ldots & x_n^{n-1} \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

where $f(X) = c_0 + c_1 X + \cdots + c_{n-1} X^{n-1}$.

Needed in our 1989 sparse polynomial multiplication algorithm [see also J. van der Hoeven, Proc. ISSAC 2004] and Shoup's polynomial factorization algorithm.

Direct solution: $V^{Tr} \cdot V = \left[\sum_k x_k^{i+j}\right]_{i,j} = H$ is Hankel, so

$$(V^{Tr} \cdot V)V^{-1}y = H(V^{-1}y) = b.$$

Uses interpolation $V^{-1}y$,
plain power sums via Newton's identities,
and polynomial multiplication.

Multipoint evaluation

Let $n = 2^m$:
Compute

$$
\begin{aligned}
1:&\ (X-x_1)(X-x_2),\ (X-x_3)(X-x_4),\ldots\\
2:&\ (X-x_1)\cdots(X-x_4),\ (X-x_5)\cdots(X-x_8),\ldots\\
&\ \vdots\\
m-1:&\ (X-x_1)\cdots(X-x_{n/2}),\ (X-x_{n/2+1})\cdots(X-x_n)
\end{aligned}
$$

then

$$
\begin{aligned}
1:&\ f(X) \bmod (X-x_1)\cdots(X-x_{n/2}),\\
&\qquad\qquad f(X) \bmod (X-x_{n/2+1})\cdots(X-x_n)\\
&\ \vdots\\
m-2:&\ f(X) \bmod (X-x_1)\cdots(X-x_4),\\
&\qquad\qquad f(X) \bmod (X-x_5)\cdots(X-x_8),\ldots\\
m-1:&\ f(X) \bmod (X-x_1)(X-x_2),\ f(X) \bmod (X-x_3)(X-x_4),\ldots\\
m:&\ f(x_1) = f(X) \bmod (X-x_1),\ f(x_2) = f(X) \bmod (X-x_2),\ldots
\end{aligned}
$$

Complexity is parameterized by polynomial multiplication/division algorithm:

$$O(M(n)\log n) \text{ arithmetic operations}$$

$M(n) = O(n^2)$ classical for small n
$M(n) = O(n^{1.59})$ [Karatsuba]
$M(n) = O(n(\log n)(\log\log n))$ [Cantor and Kaltofen]
$M(n) = O(n(\log n))$ with primitive roots

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Practically efficiently reverse all those algorithms [Bostan, Lecerf and Schost, Proc. ISSAC 2003]

Ostrowski, Wolin, and Borisow (1971) automatic differentiation (reverse mode)

Consider straight-line program

$$\begin{aligned} v_j &\leftarrow x_j, & 1 \le j \le n, \\ v_i &\leftarrow v_{L_i} \circ_i v_{R_i}, & n+1 \le i \le n+s. \end{aligned}$$

Compute all $\partial_{x_j}(v_{n+s})$ by unravelling from the back:

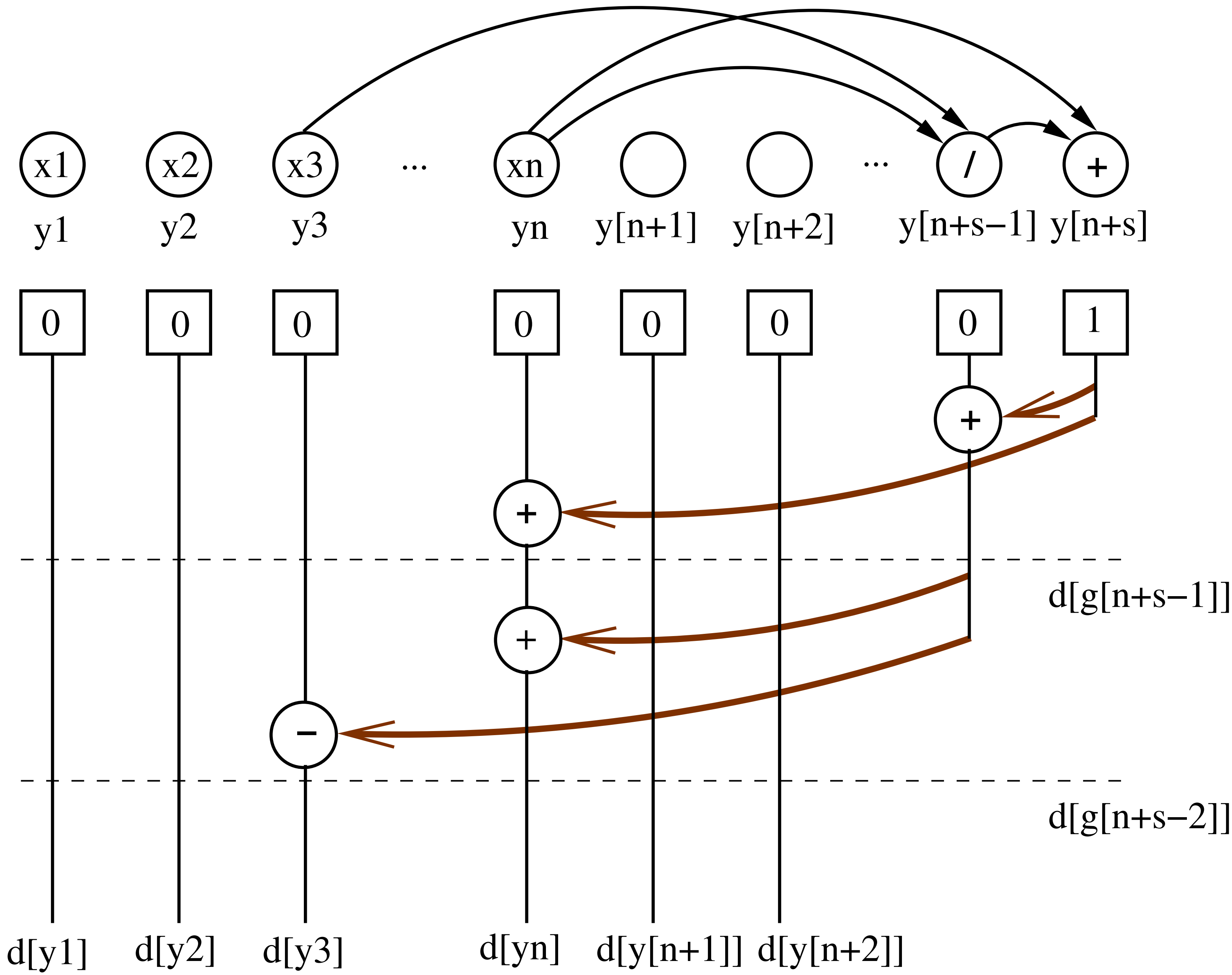
$$g_{i-1}(y_1,\ldots,y_{i-1}) := g_i(y_1,\ldots,y_{i-1},h_i(y_{L_i},y_{R_i})), i = n+s,...,n+1.$$

By chain rule for $j \ne L_i, j \ne R_i$

$$(\partial_{y_j} g_{i-1})(y_1,\ldots,y_{i-1}) = (\partial_{y_j} g_i)(y_1,\ldots,y_{i-1},h_i(y_{L_i},y_{R_i}))$$

and for $j = L_i$ or $j = R_i$

$$\begin{aligned} (\partial_{y_j} g_{i-1})(y_1,\ldots,y_{i-1}) &= (\partial_{y_j} g_i)(y_1,\ldots,y_{i-1},h_i(y_{L_i},y_{R_i})) \\ &+ (\partial_{y_i} g_i)(y_1,\ldots,y_{i-1},h_i(y_{L_i},y_{R_i})) \cdot (\partial_{y_j} h_i)(y_{L_i},y_{R_i}). \end{aligned}$$


x1
x2
x3
...
xn
...
/
+
y1
y2
y3
yn
y[n+1]
y[n+2]
y[n+s−1]
y[n+s]
0
0
0
0
0
0
0
1
+
+
+
−
d[g[n+s−1]]
d[g[n+s−2]]
d[y1]
d[y2]
d[y3]
d[yn]
d[y[n+1]]
d[y[n+2]]

Ostrowski et al. ’71: Size (= sequential time) of circuit for $\partial f/\partial x_i$
$\leq 4s$

Kaltofen & Singer ’91: Depth (= parallel time) of circuit for $\partial f/\partial x_i$
$= O(\text{depth of circuit for } f)$.

Transformation is numerically stable.

Inverted transposition principle by automatic differentiation

The problems $A^{-1} \cdot b$ and $(A^{Tr})^{-1} \cdot b$, given $A \in \mathbb{F}^{n \times n}$ non-singular and $b \in \mathbb{F}^n$, have the same asymptotic circuit complexity: Let

$$f(x_1, \ldots, x_n) = \left(\begin{bmatrix} x_1 & \ldots & x_n \end{bmatrix} \cdot (A^{Tr})^{-1} \right) \cdot b \in \mathbb{F}[x_1, \ldots, x_n].$$

Then

$$\begin{bmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_n} f \end{bmatrix} = (A^{Tr})^{-1} b.$$

Note: Transposition principle may not apply due to divisions.

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Used for:

- $(\text{Vandermonde}^{Tr})^{-1} \cdot b$ for sparse polynomial interpolation (Kaltofen and Lakshman '88).
- Constant improvements to interpolation via the transposed algorithm [Bostan, Lecerf, Schost '03].

Reduction: Matrix Inverse $\preccurlyeq$ Determinant (Baur, Strassen '83)

Consider a circuit for the determinant,

$$f(a_{1,1},\ldots,a_{n,n}) = \det\left(\begin{bmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} \end{bmatrix}\right).$$

Then

$$(-1)^{i+j}\frac{\partial f}{\partial a_{j,i}} = \det(A)(A^{-1})_{i,j}.$$

$\Longrightarrow$ Circuit for partials computes adjoint matrix. Used for:

- Division-free computation of adjoint of A in $O(n^{2.69})$ arithmetic operations (Kaltofen and Villard '01/'04).

Villard 2003: automatic differentiation does not preserve bit complexity

$x^T yc$ where x, y are vectors with constant entries,
c a large constant

takes $O(n + \log |c|)$ bit operations,
yc takes $O(n \log |c|)$ bit operations

My ECCAD'98 open problem 6

Let $\sigma \in \mathbb{F}[\alpha,\beta]/(f,g)$ where $f(\alpha,\beta)=0$ and $g(\beta)=0$.
E.g., $\sigma = \sqrt{1+\sqrt{2}}-\sqrt{2} = \alpha-\beta$, $f=\alpha^2-\beta-1$, and $g=\beta^2-2$.

Task: Compute the minimum polynomial $h(\sigma)=0$:

$$h(x) = x^m - c_{m-1}x^{m-1} - \cdots - c_0 \in \mathbb{F}[x], \quad m \le \deg(f)\cdot\deg(g)$$

The coefficient vectors $\overrightarrow{\sigma^i}$ of σ^i mod $(f(\alpha,\beta),g(\beta))$ satisfy

$$\forall\, j \ge 0\colon\ \overrightarrow{\sigma^{m+j}} = c_{m-1}\overrightarrow{\sigma^{m-1+j}} + \cdots + c_0\overrightarrow{\sigma^{j}}$$

Any non-trivial linear projection $\mathcal{L}(\overrightarrow{\sigma^i})$ preserves the linear recursion because h is irreducible.

Power Projections = Transposed Modular Polyn Composition

Linear projections of powers

$$\left[\mathcal{L}(\overrightarrow{\sigma^0})\ \ \mathcal{L}(\overrightarrow{\sigma^1})\mathcal{L}(\overrightarrow{\sigma^2})\ \ \ldots\right] = \left[u_0\ \ u_1\ \ \ldots\ \ u_{n-1}\right] \cdot \underbrace{\left[\overrightarrow{\sigma^0}\ \middle|\ \overrightarrow{\sigma^1}\ \middle|\ \overrightarrow{\sigma^2}\ \middle|\ \ldots\right]}_{A}$$

Modular polynomial composition

$$w(z) = w_0 + w_1 z + w_2 z^2 + \cdots \longmapsto w(\sigma) \bmod (f(\alpha,\beta), g(\beta))$$

$$\overrightarrow{w(\sigma)} = \underbrace{\left[\overrightarrow{\sigma^0}\ \middle|\ \overrightarrow{\sigma^1}\ \middle|\ \overrightarrow{\sigma^2}\ \middle|\ \ldots\right]}_{A} \cdot \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ \vdots \end{bmatrix}$$

By Tellegen's Principle [1960] the problems can be solved equally fast

Transposed Modular Polynomial Multiplication in NTL

1. $T_1 \leftarrow \mathrm{FFT}^{-1}(\mathrm{RED}_k(g))$
2. $T_2 \leftarrow T_1 \cdot S_2$
3. $v \leftarrow -\mathrm{CRT}_{0\ldots n-2}(\mathrm{FFT}(T_2))$
4. $T_2 \leftarrow \mathrm{FFT}^{-1}(\mathrm{RED}_{k+1}(x^{n-1} \cdot v))$
5. $T_2 \leftarrow T_2 \cdot S_3$
6. $T_1 \leftarrow T_1 \cdot S_4$
7. Replace T_1 by the 2^{k+1}-point residue table whose j-th column $(0 \leq j < 2^{k+1})$ is 0 if j is odd, and is column number $j/2$ of T_1 if j is even.
8. $T_2 \leftarrow T_2 + T_1$
9. $u \leftarrow \mathrm{CRT}_{0\ldots n-1}(\mathrm{FFT}(T_2))$

"we offer no other proof of correctness other than the validity of this transformation technique (and the fact that it does indeed work in practice)" [Shoup 1994]

Open Problem 6

With inputs $A \in \mathbb{F}^{m \times n}$ and $y \in \mathbb{F}^n$ you are given ***an algorithm*** *for $A \cdot y$ that uses $T(m,n)$ arithmetic field operations and $S(m,n)$ auxiliary space.*

Show how to construct ***an algorithm*** *for $A^T \cdot z$ where $z \in \mathbb{F}^m$ that uses $O(T(m,n))$ time and $O(S(m,n))$ space.*

Your construction must be applicable to practical problems.

Pebble game by J. C. Gilbert, G. Le Vey, and J. Masse '91

When computing $v_i \leftarrow v_{L_i} \circ_i v_{R_i}$, values of v_{L_i} and v_{R_i} must be stored in memory or recomputed.
$\longrightarrow$ not clear how to use the same number of "pebbles" for reverse program.

Use inplace operations: `x += y;`
Pebble for one operand can be used for result, which is reversible.

Not clear how to reverse the address computation (space uniformity of reverse circuit)