On the complexity of factoring sparse polynomials

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Supersparse (lacunary) polynomials

The supersparse polynomial

$$f(X_1,\ldots,X_n) = \sum_{i=1}^{t} c_i X_1^{\alpha_{i,1}} \cdots X_n^{\alpha_{i,n}}$$

is input by a list of its coefficients and corresponding term degree vectors.

$$\mathrm{size}(f) = \sum_{i=1}^{t} \Big(\mathrm{size}(c_i) + \lceil \log_2(\alpha_{i,1} \cdots \alpha_{i,n} + 2) \rceil \Big)$$

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Term degrees can be very high, e.g., $\geq 2^{500}$
Over $\mathbb{Z}_p$: evaluate by repeated squaring
Over $\mathbb{Q}$: cannot evaluate in polynomial-time except for $X_i=0,\pm 1$

Easy problems for supersparse polynomials $f = \sum_i c_i X^{\alpha_i} \in \mathbb{Z}[z]$

Cucker, Koiran, Smale 1998: Compute root $a \in \mathbb{Z}$: $f(a) = 0$.

Gap idea: if $f(a) = 0, a \neq \pm 1$ then $g_1(a) = \cdots = g_s(a) = 0$
where $f(X) = \sum_j g_j(X) X^{\alpha_j}$ and $\alpha_{j+1} - \alpha_j - \deg(g_j) \geq \chi$.

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Write $f(X) = \underbrace{g(X)}_{\deg(g) \leq k} + X^u h(X), \quad \|f\|_1 = |c_1| + \cdots + |c_t|$.

For $a \neq \pm 1$, $h(a) \neq 0$:
$$\begin{aligned} |g(a)| &< \|f\|_1 \cdot |a|^k \\ |a^u h(a)| &\geq |a|^u \end{aligned}$$

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$$u - k \geq \chi = \log_2 \|f\|_1 \Longrightarrow |a|^u \geq 2^\chi \cdot |a|^k \geq \|f\|_1 \cdot |a|^k \Longrightarrow f(a) \neq 0.$$

Polynomial time root-finder uses the fact that for

$$g_j(X) = c_1 + c_2 x^{\beta_2} + \cdots + c_s x^{\beta_s}, \quad \beta_i - \beta_{i-1} < \chi, \quad s \leq t$$

we have

$$\beta_i \leq (i-1)(\chi - 1),$$

so

$$\deg(g_j) \leq (t-1)(\chi - 1)$$

Generalization by H. W. Lenstra, Jr. 1999

Input: $\varphi(\zeta) \in \mathbb{Z}[\zeta]$ monic irred.; let $K = \mathbb{Q}[\zeta]/(\varphi(\zeta))$
a supersparse $f(X) = \sum_{i=1}^{t} c_i X^{\alpha_i} \in K[X]$
a factor degree bound d

Output: a list of all irreducible factors of f over K of degree $\leq d$
and their multiplicities (which is $\leq t$ except for X)

Let $D = d \cdot \deg(\varphi)$
There are at most $O(t^2 \cdot 2^D \cdot D \cdot \log(Dt))$ factors of degree $\leq d$

Bit complexity is $\left(\underline{t + \log(\deg f)} + \log\|f\| + \log\|\varphi\|\right)^{O(D)}$

Special case $\varphi = \zeta - 1, d = D = 1$: Algorithm finds all rational roots in polynomial-time.

Linear and quadratic bivariate factors

Input: a supersparse $f(X,Y) = \sum_{i=1}^{t} c_i X^{\alpha_i} Y^{\beta_i} \in \mathbb{Z}[X,Y]$ that is monic in X;
an error probability $\varepsilon = 1/2^l$

Output: a list of polynomials $g_j(X,Y)$ with $\deg_X(g_j) \le 2$ and $\deg_Y(g_j) \le 2$;
a list of corresponding multiplicities.

The g_j are with probability $\ge 1-\varepsilon$ all irreducible factors of f over $\mathbb{Q}$ of degree ≤ 2 together with their true multiplicities.

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Bit complexity: $\big(t + \log(\deg f) + \log \|f\| + \log 1/\varepsilon\big)^{O(1)}$

Algorithm

Step 0: compute all factors of f that are in $\mathbb{Q}[Y]$ by Lenstra's method on the coefficients of X^{α_i}

Step 1: compute linear and quadratic factors in $\mathbb{Q}[X]$ of $f(X,0)$, $f(X,1)$ and $f(X,-1)$ by Lenstra's method

Step 2: interpolate all factor combinations;
Test if $g(X,Y)$ divides $f(X,Y)$ by

$0 \equiv f(X,a) \bmod (g(X,a),p)$ where $a \in \mathbb{Z}$, p prime are random

Leading coefficient problem

If the leading (trailing) coefficient in X does not vanish for $Y = 0, \pm 1$, then one can impose *a factor* of the leading (trailing) coefficient on g.

Cannot interpolate factors of $\sum_i (X^{2d_i} - 1)(Y^{2e_i} - 1) f_i(X,Y)$

But we can compute all factors $Y - aX - b$ of all supersparse polynomials deterministically

Generalized gap theorem

$$Y - aX - b \text{ divides } f(X,Y) \iff 0 = \sum_{i=1}^{t} c_i X^{\alpha_i} (aX+b)^{\beta_i}$$

Write $f(X,Y) = g(X,Y) + Y^u h(X,Y)$ with $\deg_Y(g) \leq k$.
If

$$u - k > \chi = 15.45 \cdot \log_2(t \cdot \operatorname{Height} f)$$

where

$$\operatorname{Height} f = \max_i |c_i| \quad \text{provided} \quad \operatorname{GCD}_i(c_i) = 1,$$

then for rational $(a,b) \neq (0,0), (\pm 1, 0), (0, \pm 1)$

$$f(X, aX+b) = 0 \Longrightarrow g(X, aX+b) = 0 \text{ and } h(X, aX+b) = 0.$$

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Note: $1 \big/ \log_2 \min_{d \geq 5} (1 + \cos \frac{2\pi}{5})^{\frac{\lfloor d/5 \rfloor}{d-1}} < 15.45$

Key idea in proof

Assume $g(X, aX+b) = -(aX+b)^u h(X, aX+b) \neq 0$.
Evaluate at roots of unity θ and use absolute values v and Weil height H:

$$\max(1, |a+b\theta|_\nu)^{u-k} \cdot |g(\theta, a+b\theta)|_\nu \leq \max(1, |t|_\nu) \cdot |f|_\nu \cdot |a+b\theta|_\nu^u.$$

Taking a fractional power $d_\nu/[K:\mathbb{Q}]$ and product over all v, using the product formula $\prod_\nu |\eta|_\nu^{d_\nu} = 1$ $(\eta \neq 0)$,

$$H(a+b\theta)^{u-k} \leq t \cdot H(f).$$

The Bogomolov property for algebraic number fields implies that

$$H(a+b\theta) > 1.045, \quad (a,b) \neq (0,0), (\pm 1, 0), (0, \pm 1).$$

Polynomial time algorithm for unknown a, b uses the facts

1. gap is independent of $a, b \in \mathbb{Q}$

2. first splits into segments $g_j(X, Y)$ with $\deg_Y(g_j) \le (t-1)(\chi-1)$

3. switches roles of X, Y in each g_j and splits into segments $g_{j,\ell}$ with $\deg_X(g_{j,\ell}) \le (t-1)(\chi-1)$

Generalization via Bogomolov properties of algebraic numbers

Theorem [cf. Amoroso and Zannier 2000]
Let L be a cyclotomic, hence Abelian extension of $\mathbb{Q}$.
For any algebraic $\eta \neq 0$ that is not a root of unity

$$\log H(\eta) \geq \frac{C_1}{D}\left(\frac{\log(2D)}{\log\log(5D)}\right)^{-13},$$

where $C_1 > 0$ and $D = [L(\eta) : L]$.

Thus, for coefficients in $\mathbb{Q}(\zeta)$ we have a gap bound

$$[\mathbb{Q}(\zeta) : \mathbb{Q}]^{1+o(1)} \cdot \log(t \cdot \mathrm{Height}(f)).$$

Further generalizations via Lang conjecture (Faltings's theorem, etc.).

Hard problems for supersparse polynomials in $K[X,Y]$

Theorem
The set of all monic (in X) irreducible supersparse polynomials in $K[X,Y]$ is co-NP-hard for $K = \mathbb{Q}$ and $K = \mathbb{F}_q$ for all p and all sufficiently large $q = p^k$, via randomized reduction.

Corollary
Suppose we have a Monte Carlo polynomial-time irreducibility test for monic supersparse polynomials in $\mathbb{F}_{2^k}[x,y]$ (for sufficiently large k).
Then large integers can be factored in Las Vegas polynomial-time.

Supersparse integers

$2^{2^{2478782}}+1$ is divisible by $6\cdot 2^{2478784}+1$ [Cosgrave et al. 2003]

The factors of $2^{2^n}+1$ are primes of the form $k\cdot 2^{n+2}+1$
$641 = 5\cdot 2^7+1$ divides $2^{2^5}+1$ [Euler 1732]