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Complexity Theory in the Service of Algorithm Design

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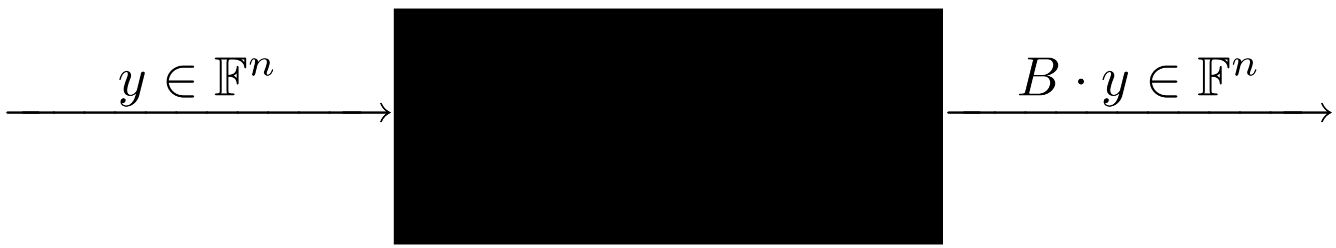
Outline

- **Wiedemann's sparse linear system solver**
  - Coordinate recurrences
  - More applications of the transposition principle
- **Reverse mode of automatic differentiation**
  - Transposition principle by derivatives
  - More applications
- **Polynomial factorization**
  - Berlekamp's polynomial factorization algorithm
  - use of the Wiedemann method
  - new baby step/giant step algorithm

A “black box” matrix

is an efficient **procedure** with the specifications



$B \in \mathbb{F}^{n \times n}$

$\mathbb{F}$ an arbitrary field

i.e., the matrix is not stored explicitly, its structure is unknown.

Main algorithmic problem: How to efficiently solve a linear system with a black box coefficient matrix?

Idea for Wiedemann's algorithm

$B \in \mathbb{F}^{n\times n}$, $\mathbb{F}$ a (possibly finite) field

$\phi^B(\lambda) = c'_0 + c'_1\lambda + \cdots + c'_m\lambda^m \in \mathbb{F}[\lambda]$ **minimum polynomial** of B:

$\forall\, u, v \in \mathbb{F}^n$: $\forall\, j \geq 0$: $u^{\mathrm{tr}} B^j \phi^B(B) v = 0$

$$\Updownarrow$$

$$c'_0 \cdot \underbrace{u^{\mathrm{tr}} B^j v}_{a_j} + c'_1 \cdot \underbrace{u^{\mathrm{tr}} B^{j+1} v}_{a_{j+1}} + \cdots + c'_m \cdot \underbrace{u^{\mathrm{tr}} B^{j+m} v}_{a_{j+m}} = 0$$

$$\Updownarrow$$

$\{a_0, a_1, a_2, \ldots\}$ is generated by a linear recursion

Theorem (Wiedemann 1986): *For random* $u, v \in \mathbb{F}^n$, *a linear generator for* $\{a_0, a_1, a_2, \ldots\}$ *is one for* $\{I, B, B^2, \ldots\}$.

$$\forall\, j \geq 0\colon c_0 a_j + c_1 a_{j+1} + \cdots + c_d a_{j+d} = 0$$

$$\Big\Downarrow \text{(with high probability)}$$

$$c_0 B^j v + c_1 B^{j+1} v + \cdots + c_d B^{j+d} v = 0$$

$$\Big\Downarrow \text{(with high probability)}$$

$$c_0 B^j + c_1 B^{j+1} + \cdots + c_d B^{j+d} = 0$$

that is, $\phi^B(\lambda)$ divides $c_0 + c_1\lambda + \cdots + c_m\lambda^m$

Algorithm *Homogeneous Wiedemann*

Input: $B \in \mathbb{F}^{n\times n}$ singular

Output: $w \neq \mathbf{0}$ such that $Bw = \mathbf{0}$

Step W1: Pick random $u, v \in \mathbb{F}^n$; $\quad b \leftarrow Bv$;
for $i \leftarrow 0$ **to** $2n-1$ **do** $a_i \leftarrow u^{\mathrm{tr}} B^i b$.
(Requires $2n$ black box calls.)

Step W2: Compute a linear recurrence generator for $\{a_i\}$,
$c_\ell \lambda^\ell + c_{\ell+1}\lambda^{\ell+1} + \cdots + c_d \lambda^d, \quad \ell \geq 0, d \leq n, c_\ell \neq 0,$
by the Berlekamp/Massey algorithm.

Step W3: $\widehat{w} \leftarrow c_\ell v + c_{\ell+1} Bv + \cdots + c_d B^{d-\ell} v$;
(With high probability $\widehat{w} \neq 0$ and $B^{\ell+1}\widehat{w} = 0$.)
Compute first k with $B^k \widehat{w} = 0$; **return** $w \leftarrow B^{k-1}\widehat{w}$.

Steps W1 and W3 have the same computational complexity

$$u^{\mathrm{tr}} \cdot [\, v \mid Bv \mid B^2 v \mid \ldots \mid B^{2n} v \,] = [\, a_{-1} \quad a_0 \quad a_1 \quad \ldots \quad a_{2n-1} \,]$$

$$[\, v \mid Bv \mid B^2 v \mid \ldots \mid B^{2n} v \,] \cdot \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2n} \end{bmatrix} = w$$

Fact: $X \cdot y$ and $X^{\mathrm{tr}} \cdot z$ have the same computational complexity [Kaminski *et al.*, 1988].