Approximate Factorization of Complex Multivariate Polynomials

Erich Kaltofen

North Carolina State University

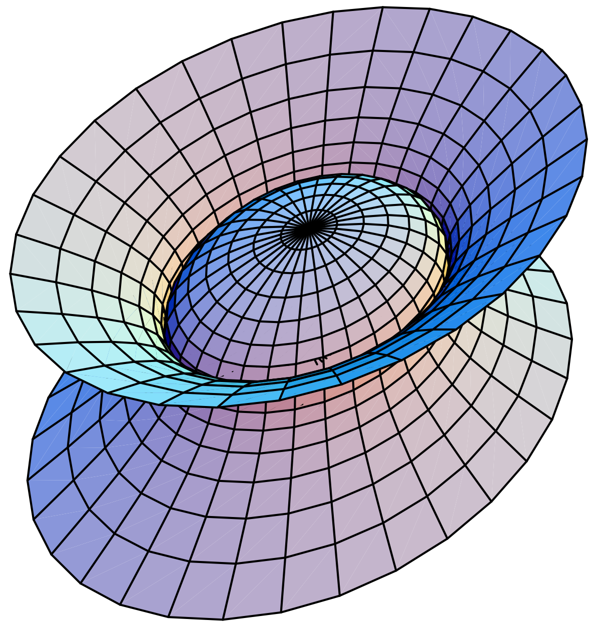
http://www.kaltofen.us

Joint work with Shuhong Gao, John May, Zhengfeng Yang, and Lihong Zhi

May and Yang received the *ACM SIGSAM's ISSAC 2004 Distinguished Student Author Award* for this work

Factorization of noisy polynomials over the complex numbers [my '98 "Challenges"]

$$81x^4 + 16y^4 - 648z^4 + 72x^2y^2 - 648x^2 - 288y^2 + 1296 = 0$$



$$(9x^2 + 4y^2 + 18\sqrt{2}z^2 - 36)(9x^2 + 4y^2 - 18\sqrt{2}z^2 - 36) = 0$$

$$81x^4 + 16y^4 - 648.003z^4 + 72x^2y^2 + .002x^2z^2 + .001y^2z^2$$
$$- 648x^2 - 288y^2 - .007z^2 + 1296 = 0$$

Conclusion on my exact algorithm [JSC 1(1)’85]

“D. Izraelevitz at Massachusetts Institute of Technology has already implemented a version of algorithm 1 using complex floating point arithmetic. Early experiments indicate that the linear systems computed in step (L) tend to be ***numerically ill-conditioned.*** *How to overcome this numerical problem is an important question which we will investigate.”*

The Approximate Factorization Problem [LATIN '94]

Given $f \in \mathbb{C}[x,y]$ irreducible, find $\tilde{f} \in \mathbb{C}[x,y]$ s.t. $\deg \tilde{f} \leq \deg f$, $\tilde{f}$ factors, and $\| f - \tilde{f} \|$ is minimal.

The Approximate Factorization Problem [LATIN '94]

Given $f \in \mathbb{C}[x,y]$ irreducible, find $\tilde{f} \in \mathbb{C}[x,y]$ s.t. $\deg \tilde{f} \leq \deg f$, $\tilde{f}$ factors, and $\|f - \tilde{f}\|$ is minimal.

Problem depends on choice of norm $\|\cdot\|$, and
notion of degree.

We use 2-norm, and multi-degree: $\operatorname{mdeg} f = (\deg_x f, \deg_y f)$

The Approximate Factorization Problem [LATIN '94]

Given $f \in \mathbb{C}[x,y]$ irreducible, find $\tilde{f} \in \mathbb{C}[x,y]$ s.t. $\deg \tilde{f} \le \deg f$, $\tilde{f}$ factors, and $\| f - \tilde{f} \|$ is minimal.

Problem depends on choice of norm $\| \cdot \|$, and
notion of degree.

We use 2-norm, and multi-degree: $\text{mdeg } f = (\deg_x f, \deg_y f)$

Degree bound is important:
$(1 + \delta x) f$ is reducible but for $\delta < \varepsilon / \| f \|$,

$$\|(1+\delta x) f - f\| = \|\delta x f\| = \delta \|f\| < \varepsilon$$

State of the Approximate Factorization

- No polynomial time algorithm (except for constant degree factors [Hitz, Kaltofen, Lakshman '99])

State of the Approximate Factorization

- No polynomial time algorithm (except for constant degree factors [Hitz, Kaltofen, Lakshman ’99])

- Several algorithms and heuristics to find a nearby factorizable $\hat{f}$ if f is “nearly factorizable” [Corless et al. ’01 & ’02, Galligo and Rupprecht ’01, Galligo and Watt ’97, Huang et al. ’00, Sasaki ’01,...]

State of the Approximate Factorization

- No polynomial time algorithm (except for constant degree factors [Hitz, Kaltofen, Lakshman '99])

- Several algorithms and heuristics to find a nearby factorizable $\hat{f}$ if f is "nearly factorizable" [Corless et al. '01 & '02, Galligo and Rupprecht '01, Galligo and Watt '97, Huang et al. '00, Sasaki '01,...]

- There are lower bounds for $\min \|f - \tilde{f}\|$ ("irreducibility radius") [Kaltofen and May ISSAC 2003]

Our ISSAC’04 Results

- A new practical algorithm to compute approximate multivariate GCDs

Our ISSAC'04 Results

- A new practical algorithm to compute approximate multivariate GCDs

- A practical algorithm to find the factorization of a nearby factorizable polynomial given any f

  especially "noisy" f:
  Given $f = f_1 f_2 + f_{\text{noise}}$,
  we find $\bar{f}_1, \bar{f}_2$ s.t. $\| f_1 f_2 - \bar{f}_1 \bar{f}_2 \| \approx \| f_{\text{noise}} \|$

  even for large noise: $\| f_{\text{noise}} \| / \| f \| \geq 10^{-3}$

Maple Demonstration

Ruppert's Theorem

$f \in \mathbb{K}[x,y]$, mdeg $f = (m,n)$

$\mathbb{K}$ is a field, algebraically closed, and characteristic 0

Theorem. f is reducible $\iff \exists g,h \in \mathbb{K}[x,y]$, non-zero,

$$\frac{\partial}{\partial y}\frac{g}{f} - \frac{\partial}{\partial x}\frac{h}{f} = 0$$

$$\text{mdeg}\, g \leq (m-2,n)\,,\ \text{mdeg}\, h \leq (m,n-1)$$

Ruppert's Theorem

$f \in \mathbb{K}[x,y]$, mdeg $f = (m,n)$

$\mathbb{K}$ is a field, algebraically closed, and characteristic 0

Theorem. f is reducible $\iff \exists g,h \in \mathbb{K}[x,y]$, non-zero,

$$\frac{\partial}{\partial y}\frac{g}{f} - \frac{\partial}{\partial x}\frac{h}{f} = 0$$

$$\text{mdeg } g \leq (m-2,n)\,, \text{mdeg } h \leq (m,n-1)$$

PDE $\rightsquigarrow$ linear system in the coefficients of g and h

Gao's PDE based Factorizer

Change degree bound: $\mathrm{mdeg}\, g \leq (m-1, n), \mathrm{mdeg}\, h \leq (m, n-1)$

so that: # linearly indep. solutions to the PDE = # factors of f

Require square-freeness: $\mathrm{GCD}(f, \frac{\partial f}{\partial x}) = 1$

Gao's PDE based Factorizer

Change degree bound: $\operatorname{mdeg} g \leq (m-1,n), \operatorname{mdeg} h \leq (m,n-1)$

so that: # linearly indep. solutions to the PDE = # factors of f

Require square-freeness: $\mathrm{GCD}(f, \frac{\partial f}{\partial x}) = 1$

Let

$$G = \mathrm{Span}_{\mathbb{C}}\{g \mid [g,h] \text{ is a solution to the PDE}\}.$$

Any solution $g \in G$ satisfies $g = \sum_{i=1}^{r} \lambda_i \frac{\partial f_i}{\partial x} \frac{f}{f_i}$ with $\lambda_i \in \mathbb{C}$, so

$$f = f_1 \cdots f_r = \prod_{\lambda \in \mathbb{C}} \gcd(f, g - \lambda \frac{\partial f}{\partial x})$$

(f_i the distinct irreducible factors of f)

With high probability $\exists$ distinct λ_i s.t. $f_i = \gcd(f, g - \lambda_i \frac{\partial f}{\partial x})$

Gao's PDE based Factorizer

Algorithm

Input: $f \in \mathbb{K}[x,y]$, $\mathbb{K} \subseteq \mathbb{C}$
Output: $f_1, \ldots, f_r \in \mathbb{C}[x,y]$

1. Find a basis for the linear space G, and choose a random element $g \in G$.

2. Compute the polynomial $E_g = \prod_i (z - \lambda_i)$ via an eigenvalue formulation
   If E_g not squarefree, choose a new g

3. Compute the factors $f_i = \gcd(f, g - \lambda_i \frac{\partial f}{\partial x})$ in $\mathbb{K}(\lambda_i)$.

In exact arithmetic the extention field $\mathbb{K}(\lambda_i)$ is found via univariate factorization.

Adapting to the Approximate Case

The following must be solved to create an approximate factorizer from Gao's algorithm:

1. Computing approximate GCDs of bivariate polynomials;
2. Determining the numerical dimension of G, and computing an approximate solution g;
3. Computing a g s.t. the polynomial E_g has no clusters of roots.

Determining the Number of Approximate Factors

Let $\mathrm{Rup}(f)$ be the matrix from Gao's algorithm
Recall:

$$\#\text{ of factors of } f = \mathrm{Nullity}(\mathrm{Rup}(f))$$

Determining the Number of Approximate Factors

Let $\mathrm{Rup}(f)$ be the matrix from Gao's algorithm
Recall:

$$\#\text{ of factors of } f = \mathrm{Nullity}(\mathrm{Rup}(f))$$

$\mathrm{Rup}(f)$ has nullity r if
$\sigma_m \geq \ldots \geq \sigma_{r+1} \neq 0$ and $\sigma_r = \ldots = \sigma_1 = 0$.

Say $\mathrm{Rup}(f)$ has nullity r with tolerance ε if:

$$\sigma_m \geq \ldots \geq \sigma_{r+1} > \varepsilon \geq \sigma_r \geq \ldots \geq \sigma_1$$

Find a "best" ε from the largest gap
choose $\varepsilon = \sigma_r$ s.t. σ_{r+1}/σ_r is maximal

Determining the Number of Approximate Factors

If f is irreducible
largest gap in the sing. values of $\mathrm{Rup}(f)$ $\rightsquigarrow$ # of approx. factors

Recall:

$$G = \mathrm{Span}_{\mathbb{C}}\{g \mid [g,h] \in \mathrm{Nullspace}(\mathrm{Rup}(f))\}$$

If r is position of the largest gap in the sing. values of $\mathrm{Rup}(f)$,
approx. version of G is Span of last r sing. vectors of $\mathrm{Rup}(f)$

Approximate Factorization

Input: $f \in \mathbb{C}[x,y]$ abs. irreducible, approx. square-free
Output: $f_1, \ldots, f_r$ approx. factors of f, and c

1. Compute the SVD of $\mathrm{Rup}(f)$, determine r, its approximate nullity, and choose $g = \sum a_i g_i$, a random linear combination of the last r right singular vectors

2. compute E_g and its roots via an eigenvalue computation

3. For each λ_i compute the approximate GCD $f_i = \gcd(f, g - \lambda_i f)$ and find an optimal scaling: $\min_c \| f - c \prod_{i=1}^{r} f_i \|$

Approx. GCD: Generalized Sylvester Matrix

A pair $g, h \in \mathbb{K}[x, y]$ has GCD of degree at least k iff
$\exists$ non-zero solutions $u, v \in \mathbb{K}[x, y]$ to:

$$\frac{g}{h} = \frac{v}{u}, \ \mathrm{tdeg}(u) \leq \mathrm{tdeg}(h) - k, \ \mathrm{tdeg}(v) \leq \mathrm{tdeg}(g) - k$$

or

$$ug - vh = 0, \ \mathrm{tdeg}(u) \leq \mathrm{tdeg}(h) - k, \ \mathrm{tdeg}(v) \leq \mathrm{tdeg}(g) - k$$

Equation gives a linear system in the coefficients of u and v

Denote the matrix of the system $\mathrm{Syl}_k(g, h)$

Computing the Approximate GCD

Input: g and h relatively prime
Output: $d \notin \mathbb{K}$, approx. GCD of g and h

1. Find p from the largest gap in the singular values of $\mathrm{Syl}_1(g, h)$
2. Find $k \in \mathbb{Z}$ which solves $\min_k \left| p - \binom{k+2}{2} \right|$
3. Find $[\mathbf{u}, \mathbf{v}]$, the right singular vector corresponding to smallest singular value of $\mathrm{Syl}_k(g, h)$
[compute with an iterative method]
4. Find a d to minimize $\|h - d\,u\|_2^2 + \|g - d\,v\|_2^2$, using least squares ("Approximate division")

Also possible to add iterative improvement á la Zeng&Dayton'04

Notes on the Repeated Factor Case

We say f is approximately square-free if:

dist. to nearest reducible poly. $<$ dist. to nearest non-square-free poly.

Notes on the Repeated Factor Case

We say f is approximately square-free if:

dist. to nearest reducible poly. $<$ dist. to nearest non-square-free poly.

We handle the repeated factor case differently than usual:
without iterating approximate GCDs:

Compute the approximate quotient $\overline{f}$ of f and $\gcd(f, \frac{\partial f}{\partial x})$ and factor the approximately square-free kernel $\overline{f}$

Notes on the Repeated Factor Case

We say f is approximately square-free if:

dist. to nearest reducible poly. $<$ dist. to nearest non-square-free poly.

We handle the repeated factor case differently than usual: without iterating approximate GCDs:

Compute the approximate quotient $\overline{f}$ of f and $\gcd(f, \frac{\partial f}{\partial x})$ and factor the approximately square-free kernel $\overline{f}$

Determine multiplicity of approximate factors f_i by comparing the degrees of the approximate GCDs:

$$\gcd(f_i, \partial^k f / \partial x^k)$$

Table of Benchmarks

Example	$\mathrm{tdeg}(f_i)$	$\frac{\sigma_{r+1}}{\sigma_r}$	$\frac{\sigma_r}{\|R(f)\|_2}$	coeff. error	backward error	*time*(*sec*)
Nagasaka'02	$2,3$	11	10^{-3}	10^{-2}	1.08e–2	14.631
Kaltofen'00	$2,2$	10^9	10^{-10}	10^{-4}	1.02e–9	13.009
Sasaki'01	$5,5$	10^9	10^{-10}	10^{-13}	8.30e–10	5.258
Sasaki'01	$10,10$	10^5	10^{-6}	10^{-7}	1.05e–6	85.96
Corless et al'01	$7,8$	10^7	10^{-8}	10^{-9}	1.41e–8	19.628
Corless et al'02	$3,3,3$	10^8	10^{-10}	0	1.29e–9	9.234
Zeng'04	$(5)^3,3,(2)^4$	10^7	10^{-9}	10^{-10}	2.09e–7	73.52

Table of Benchmarks

Example	tdeg(f_i)	$\frac{\sigma_{r+1}}{\sigma_r}$	$\frac{\sigma_r}{\|R(f)\|_2}$	coeff. error	backward error	*time*(*sec*)
Random ($f_i \in \mathbb{Z}$)	$9,7$	486	10^{-4}	10^{-4}	2.14e–4	43.823
"	$6,6,10$	10^3	10^{-6}	10^{-5}	2.47e–4	539.67
"	$4,4,4,4,4$	273	10^{-6}	10^{-5}	1.31e–3	3098.
"	$3,3,3$	1.70	10^{-3}	10^{-1}	7.93e–1	29.25
"	$18,18$	10^4	10^{-7}	10^{-6}	3.75e–6	3173.
"	$12,7,5$	8.34	10^{-4}	10^{-3}	8.42e–3	4370.
Not Sqr Free	$5,(5)^2$	10^3	10^{-5}	10^{-5}	6.98e–5	34.28
3 variables	$5,5$	10^4	10^{-5}	10^{-5}	1.72e–5	332.99
$f_i \in \mathbb{C}$	$6,6$	10^6	10^{-8}	10^{-7}	2.97e–7	30.034

Table of Benchmarks

Example	$\mathrm{tdeg}(f_i)$	$\frac{\sigma_{r+1}}{\sigma_r}$	$\frac{\sigma_r}{\|R(f)\|_2}$	coeff. error	backward error	*time(sec)*
Random ($f_i \in \mathbb{Z}$)	$9, 7$	486	10^{-4}	10^{-4}	2.14e–4	43.823
"	$6, 6, 10$	10^3	10^{-6}	10^{-5}	2.47e–4	539.67
"	$4, 4, 4, 4, 4$	273	10^{-6}	10^{-5}	1.31e–3	3098.
"	$3, 3, 3$	1.70	10^{-3}	10^{-1}	7.93e–1	29.25
"	$18, 18$	10^4	10^{-7}	10^{-6}	3.75e–6	3173.
"	$\mathbf{12, 7, 5}$	8.34	10^{-4}	$\mathbf{10^{-3}}$	**8.42e–3**	4370.
Not Sqr Free	$5, (5)^2$	10^3	10^{-5}	10^{-5}	6.98e–5	34.28
3 variables	$5, 5$	10^4	10^{-5}	10^{-5}	1.72e–5	332.99
$f_i \in \mathbb{C}$	$6, 6$	10^6	10^{-8}	10^{-7}	2.97e–7	30.034

More than two variables: direct approach

- PDEs can be generalized to many variables

$$\frac{\partial}{\partial y_i}\frac{g}{f} - \frac{\partial}{\partial x}\frac{h_i}{f} = 0, \forall 1 \le i \le k$$

$$\deg g \le \deg f, \quad \deg h_i \le \deg f, \forall 1 \le i \le k,$$

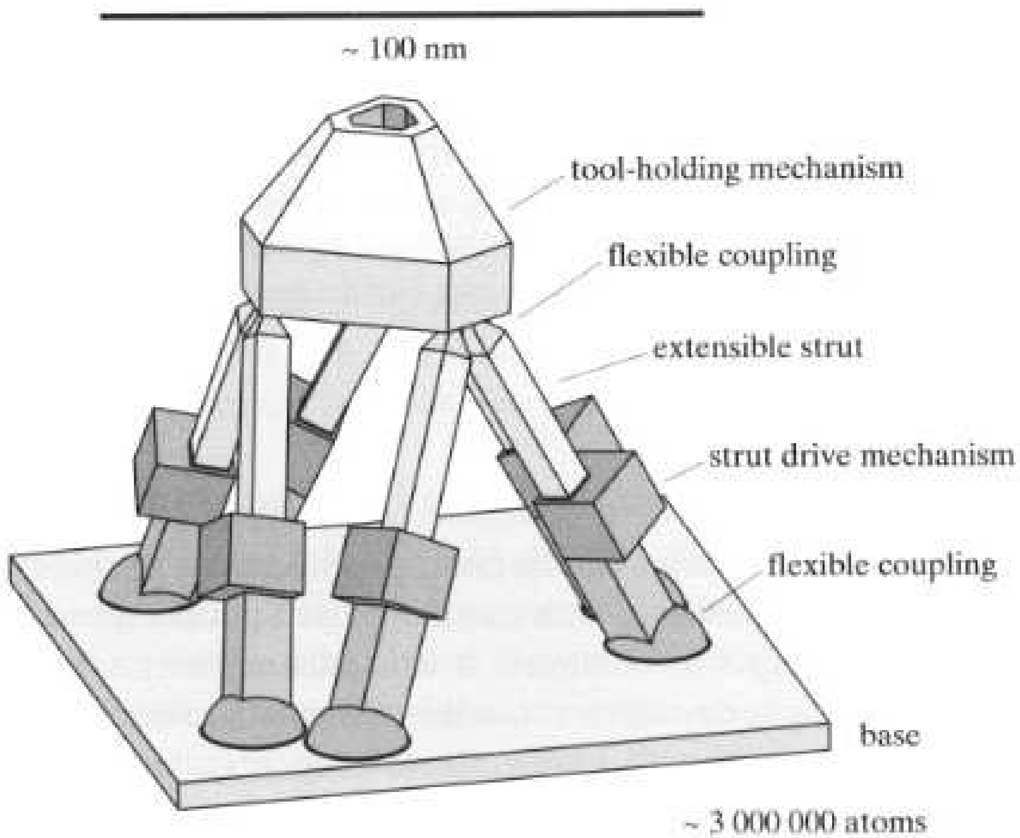
$$\deg_x g \le (\deg_x f) - 1, \quad \deg_{y_i} h_i \le (\deg_{y_i} f) - 1, \forall 1 \le i \le k.$$

More than two variables: interpolation

- Our multivariate implementation together with Wen-shin Lee's numerical *sparse* interpolation code quickly factors polynomials arising in engineering *Stewart-Gough platforms*

  Polynomials were 3 variables, factor multiplicities up to 5, coefficient error 10^{-16}, and were provided to us by Jan Verschelde

Stewart Platform Example



Drexler’s 1992 nano Stewart platform

Current Investigations

- Use *Gauss-Newton optimization* at the end to improve nearness of computed approximate answers

Current Investigations

- Use *Gauss-Newton optimization* at the end to improve nearness of computed approximate answers

- Replace SVD techniques with Structured SVD/Total Least Squares [Park et al., Chu et al.]

Current Investigations

- Use *Gauss-Newton optimization* at the end to improve nearness of computed approximate answers

- Replace SVD techniques with Structured SVD/Total Least Squares [Park et al., Chu et al.]

- More generally, use *blackbox matrix* SVD algorithms

  $\mathrm{Rup}(f) \cdot \mathbf{v}$ costs 4 polynomial multiplications

  Should make very large problems possible

Current Investigations

- Use *Gauss-Newton optimization* at the end to improve nearness of computed approximate answers
- Replace SVD techniques with Structured SVD/Total Least Squares [Park et al., Chu et al.]
- More generally, use *blackbox matrix* SVD algorithms

  $\mathrm{Rup}(f) \cdot \mathbf{v}$ costs 4 polynomial multiplications

  Should make very large problems possible
- Also need sparse interpolation for "very noisy" inputs to handle sparse multivariate problems

Code + Benchmarks at:

`http://www.mmrc.iss.ac.cn/~lzhi/Research/appfac.html`

or

`http://www.kaltofen.us`

(click on “Software”)